Volumes of projection bodies of some classes of convex bodies

Christos Saroglou
University of Crete
Department of Mathematics

Abstract

Schneider posed the problem of determining the maximal value of the affine invariant $|\Pi K|/|K|^{d-1}$, where ΠK is the projection body of the d-dimensional convex body K. Some three-dimensional conjectures of Brannen, related to Schneider's problem are confirmed. Namely, we determine the maximal value of $|\Pi K|/|K|^2$ in the class of three-dimensional zonoids, cones and double cones. Equality cases are, also, investigated. Moreover, results related to a conjecture of Petty, concerning the minimal value of the above quantity are obtained. In particular, we provide a negative answer to a question of Martini and Mustafaev.

1. Introduction

Let K be a convex body in $\mathbb{R}^d$, that contains 0 in its interior. Its support function is defined by:

$$h_K(x) = max\{ < x, y > \ | \ y \in K \ \} \ , \ x \in \mathbb{R}^d \ ,$$

where $< \cdot, \cdot >$ stands for the usual inner product in $\mathbb{R}^d$. Obviously, h_K is convex and positively homogeneous. On the other hand, it is known that any convex and positively homogeneous function is the support function of a unique convex set. Moreover, support functions are additive under Minkowski sums (i.e. vector sums). To be more specific, if L is another convex body, then $h_{K+L} = h_k + h_L$.

One may compute the support function of a line segment $[-y, y]$, centered at the origin:

$$h_{[-y,y]}(x) = | < x, y > | \ , \ x \in \mathbb{R}^d \ .$$

Thus, if Z is the Minkowski sum of the line segments $[-y_i, y_i]$, $i = 1, ..., n$,

$$h_Z(x) = \sum_{i=1}^{n} | < x, y_i > | \ , \ x \in \mathbb{R}^d \ .$$

Translations of such bodies are called zonotopes and the members of the closure of the set of all zonotopes with respect to the Hausdorff metric are called zonoids.

The projection body ΠK of K is defined by its support function:

$$h_{\Pi K}(x) = |K \mid x^{\perp}| = \frac{1}{2} \int_{S^{d-1}} | < x, y > | \ dS(K, y) \ , \ x \in S^{d-1} \ ,$$

where $|\cdot| = |\cdot|_d$ denotes the volume functional in $\mathbb{R}^d$, $K \mid x^{\perp}$ is the orthogonal projection of K onto the subspace $x^{\perp}$ orthogonal to x and $dS(K, \cdot)$ is the surface area measure of K on the unit sphere S^{d-1}.

Clearly, the projection body of a convex body is always a zonoid. The opposite is also true; any zonoid is the projection body of a convex body. In fact, the operator Π is a continuous bijection between the class of centrally symmetric convex bodies and the class of zonoids. We refer to [6] or [20] for proofs, extensions and topics related to the mentioned properties, concerning support functions and projection bodies.

One of the outstanding problems in convex geometry is the determination of the extremal values of the affine invariant $|\Pi K|/|K|^{d-1}$ (a proof of the fact that this functional is indeed affine invariant can be found in [14]). Many variants of this problem are solved. For example, Petty [15] showed that the quantity $|(\Pi K)^*| \cdot |K|^{d-1}$ is maximal if and only if K is an ellipsoid and Zhang [23] proved that it attains its minimal value if and only if K is a simplex. Here, $(\Pi K)^* = \{y | < x, y > \leq 1, \ x \in \Pi K\}$ is the polar body of ΠK. Other types of modifications have been considered by Lutwak (see e.g. [8], [9]). Unfortunately, very little progress has taken place towards the direction of the initial problem.

Petty's conjecture [15] states that the ratio $|\Pi K|/|K|^{d-1}$ is minimal if and only if K is an ellipsoid (see [7] for applications). Schneider [19] conjectured that in the class of centrally symmetric convex bodies

$$\frac{|\Pi K|}{|K|^{d-1}} \leq 2^d \ ,$$

with equality if and only if K is the affine image of cartesian products of line segments or centrally symmetric planar convex figures (the class of these bodies is identical to the class of symmetric cylinders in the three-dimensional case).

Both Petty's and Schneider's conjectures make sense only if $d \geq 3$. In fact, if K is a planar convex body, it is well known that:

$$4 \leq \frac{|\Pi K|}{|K|} \leq 6 \ , \tag{1}$$

with equality in the left if and only if K is centrally symmetric and in the right if and only if K is a triangle. These facts follow immediately from (5), (7) below and the two-dimensional Rogers-Shephard inequality [18], respectively.

Counterexamples to Schneider's conjecture were given by Brannen [2]. The same author [2] [3] conjectured that it would be true if we restricted ourselves in the class of zonoids. We prove this fact in three dimensions. We hope that the method described below can be modified to work in any dimension.

Theorem 1. If Z is a three-dimensional zonoid, then

$$|\Pi Z| \leq 2^3 |Z|^2 \ . \tag{2}$$

Equality holds if and only if Z can be written as the Minkowski sum of five line segments or as the sum of a cylinder and a line segment.

Using an inequality for the volume of polar projection bodies due to Reisner [16] [17], Theorem 1 shows that for all 3-dimensional zonoids Z we have

$$|(\Pi Z)^*| \cdot |Z|^2 \geq \frac{4}{3} \ ,$$

with equality if and only if Z is a parallelepiped. We note here that the problem of finding the minimum of the quantity $|(\Pi K)^*| \cdot |K|^{d-1}$ among centrally symmetric convex bodies still remains open. Makai and Martini [11] conjectured that this minimum is attained if and only if K is a parallelepiped.

The proof of (2) and of the characterization of equality cases will be given in Sections 3 and 4 respectively.

We also deal with two other three-dimensional classes of convex bodies. To be more specific, we determine the extremal values of $|\Pi K|/|K|^2$ in the special case, in which K is a cone or a centrally symmetric double cone.

Theorem 2. Let $K = conv(P \cup \{e_3\})$ be a three-dimensional cone, where P is a convex body in $\mathbb{R}^2 \times \{0\}$ of area 1 and $e_3 = (0, 0, 1)$. Then,

$$|\Pi K| = \frac{1}{2} + \frac{1}{4} |\Pi P| \ .$$

Corollary 1. Let K be a cone in $\mathbb{R}^3$. Then,

$$13.5 \leq \frac{|\Pi K|}{|K|^2} \leq 18 \ .$$

Equality holds in the right if and only if K is a simplex and in the left if and only if K has centrally symmetric basis.

Corollary 1 follows immediately from (1) and Theorem 2.

Corollary 2. Let K be a centrally symmetric double cone in $\mathbb{R}^3$. Then,

$$\frac{|\Pi K|}{|K|^2} = 9 \ .$$

Corollaries 1 and 2 are also conjectures of Brannen [3]. Theorem 2 and Corollary 2 will be treated in Section 5. We mention that Brannen conjectured that $|\Pi K|/|K|^2$ is maximal in the class of centrally symmetric convex bodies if and only if K is a centrally symmetric double cone and in the class of general convex bodies if and only if K is a simplex.

The Steiner symmetrization $St_\nu K$ of a d-dimensional convex body K along the direction $\nu \in S^{d-1}$ is defined to be the unique convex body with the property that for any line l parallel to ν, the line segment $l \cap St_\nu K$ is symmetric with respect to the hyperplane $\nu^\perp$ and also $|l \cap St_\nu K|_1 = |l \cap K|_1$. Martini and Mustafaev [10] asked if the inequality

$$|\Pi(St_\nu K)| \leq |\Pi K| \tag{3}$$

holds for every direction $\nu \in S^{n-1}$. It is well known that the volume of K remains unchanged under Steiner symmetrization and, furthermore, K can always be transformed to a ball after applying an appropriate sequence of Steiner symmetrizations. Thus, it is clear that Petty's conjectured inequality would follow from (3). We prove, however, that (3) is not true in general. This will be an easy application of Corollary 2.

Theorem 3. For any $d \geq 3$, there exists a convex body K and a direction $\nu \in S^{d-1}$, such that

$$|\Pi(St_\nu K)| > |\Pi K| \ .$$

Proof. We will make use of the following easy fact: If K is a convex body of volume 1 in $\mathbb{R}^{d-1}$, then

$$|\Pi(K \ \times \ [-1/2, 1/2])|_d = 2 \cdot |\Pi K|_{d-1} \ . \tag{4}$$

Note that if K is a convex body contained in the subspace orthogonal to some direction ν and I is a line segment parallel to ν, then for any unit vector w in $\nu^{\perp}$, we have

$$St_w(K+I) = (St_w K) + I \ .$$

Using this fact, (4) and an inductive argument, we conclude that we only have to construct a three-dimensional counterexample for (3).

Let C be a centered three dimensional cube of volume 1. Choose some vertex v of C and set ν to be the direction parallel to $[-v, v]$. Then, one may check that the Steiner symmetrization of C along the direction ν is a centrally symmetric double cone built on a regular hexagon, which is contained in the plane $\nu^{\perp}$. Then, by Corollary 2 we have:

$$|\Pi C| = 8 < 9 = |\Pi(St_\nu C)| \ . \ \square$$

Some further results on Petty's conjecture, involving centroid bodies and mean values of volumes of projection bodies, are included in Section 6.

2. Some basic formulas

Let $Z = \sum_{i=1}^{n} [-x_i, x_i]$ be a zonotope in $\mathbb{R}^d$. The volume of Z is given by (see [22] for proof and extensions):

$$|Z| = 2^d \sum_{\{i_1,...,i_d\} \subseteq [n]} |\ det(x_{i_1}, ..., x_{i_d})| = \frac{2^d}{d!} \sum_{i_1,...,i_d \in [n]} |det(x_{i_1}, ..., x_{i_d})\ | \ , \quad (5)$$

where $[n] := \{1, ..., n\}$.

Thus, if $F_1, ..., F_n$ are the facets of a polytope K in $\mathbb{R}^d$ with corresponding outer normal unit vectors $x_1, ..., x_n$, by the definition of ΠK and (5), we have:

$$|\Pi K| = \sum_{\{i_1,...,i_d\} \subseteq [n]} |F_{i_1}| ... |F_{i_d}| \cdot |\ det(x_{i_1}, ..., x_{i_d})\ | \quad (6)$$

Suppose, now, that $K = Z = \sum_{i=1}^{n} [-x_i, x_i]$. If we, in addition, assume that any d vectors from $x_1, ..., x_n$ are linearly independent, it can be proven that its facets are exactly (up to translation) the $(d-1)$-dimensional parallelepipeds of the form $\sum_{i=1}^{d-1} [-x_{j_i}, x_{j_i}]$, where $1 \leq j_1 < ... < j_{d-1} \leq n$. In other words, the outer unit normals to the facets of Z, multiplied by the $(d-1)$-dimensional volume of the corresponding facet, are exactly the vectors:

$$\pm\ 2^{d-1}\ x_{i_1} \wedge ... \wedge x_{i_{d-1}} \ , \ 1 \leq i_1 < ... < i_{d-1} \leq n \ ,$$

where $x_1 \wedge ... \wedge x_{d-1}$ stands for the vector product of $x_1, ..., x_{d-1}$.

Applying formula (6), we immediately obtain:

$|\Pi Z| =$

$$\frac{2^{d^2}}{((d-1)!)^d d!} \sum_{i_1,...,i_{d(d-1)} \in [n]} | \, det(x_{i_1} \wedge ... \wedge x_{i_{d-1}}, \; ... \; , x_{i_{(d-1)(d-1)+1}} \wedge ... \wedge x_{i_{d(d-1)}}) \, | \, . \tag{7}$$

It is obvious that this identity holds even if we do not assume the x_i's to be in general position.

3. Proof of the main inequality

Define the functions $S, T : (\mathbb{R}^3)^6 \to \mathbb{R}_+$ by:

$$S(x_1, ..., x_6) = \sum_{\substack{i_1,...,i_6 \in [6] \\ i_j \neq i_k \text{ for } j \neq k}} | \, det(x_{i_1}, x_{i_2}, x_{i_3}) \cdot det(x_{i_4}, x_{i_5}, x_{i_6}) \, | \, ,$$

$$T(x_1, ..., x_6) = \sum_{\substack{i_1,...,i_6 \in [6] \\ i_j \neq i_k \text{ for } j \neq k}} | \, det(x_{i_1} \wedge x_{i_2} \; , \; x_{i_3} \wedge x_{i_4} \; , \; x_{i_5} \wedge x_{i_6}) \, | \, .$$

Clearly, S and T are convex and positively homogeneous on each one of their variables. Also, it is easy to see that $S(x_1, ..., x_6) = 0$ if and only if $T(x_1, ..., x_6) = 0$. We will use the convexity property in the following form:

Lemma 3.1. Let f, g be real functions defined on an open interval (a, b) of the real line. Suppose, also, that g is strictly positive in (a, b), f is convex and g is affine. Then, the ratio f/g admits a maximum value in (a, b) if and only if f is a constant multiple of g.

Let us now rewrite (5) and (7) involving T and S. If $Z = \sum_{i=1}^n [-x_i, x_i]$ is a zonotope in $\mathbb{R}^3$, we first note that we may assume that $n \geq 6$ (by taking some of the x_i's equal to each other, if necessary). We have:

$$|\Pi Z| = \frac{2^9}{2^3 \cdot 3!} \sum_{i_1,...,i_6 \in [n]} | \, det(x_{i_1} \wedge x_{i_2} \; , \; x_{i_3} \wedge x_{i_4} \; , \; x_{i_5} \wedge x_{i_6}) \, |$$

$$= \frac{2^6}{3!} \Big(\sum_{\substack{i_1,i_2,i_3 \in [n] \\ i_j \neq i_k \text{ for } j \neq k}} T(x_{i_1}, x_{i_1}, x_{i_2}, x_{i_2}, x_{i_3}, x_{i_3})$$

$$+ \sum_{\substack{i_1,i_2,i_3,i_4 \in [n] \\ i_j \neq i_k \text{ for } j \neq k}} T(x_{i_1}, x_{i_1}, x_{i_2}, x_{i_2}, x_{i_3}, x_{i_4}) + \sum_{\substack{i_1,i_2,i_3,i_4,i_5 \in [n] \\ i_j \neq i_k \text{ for } j \neq k}} T(x_{i_1}, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5})$$

$$+ \sum_{\substack{i_1,i_2,i_3,i_4,i_5,i_6\in[n] \\ i_j\neq i_k \text{ for } j\neq k}} T(x_{i_1},x_{i_2},x_{i_3},x_{i_4},x_{i_5},x_{i_6})\Big) \ . \tag{8}$$

Similarly,

$$|Z|^2 = \Big(\frac{2^3}{3!}\Big)^2 \sum_{i_1,\dots,i_6\in[n]} |\ det(x_{i_1},x_{i_2},x_{i_3}) \cdot det(x_{i_4},x_{i_5},x_{i_6})\ |$$

$$= \frac{2^6}{(3!)^2}\Big(\sum_{\substack{i_1,i_2,i_3\in[n] \\ i_j\neq i_k \text{ for } j\neq k}} S(x_{i_1},x_{i_1},x_{i_2},x_{i_2},x_{i_3},x_{i_3})$$

$$+ \sum_{\substack{i_1,i_2,i_3,i_4\in[n] \\ i_j\neq i_k \text{ for } j\neq k}} S(x_{i_1},x_{i_1},x_{i_2},x_{i_2},x_{i_3},x_{i_4}) + \sum_{\substack{i_1,i_2,i_3,i_4,i_5\in[n] \\ i_j\neq i_k \text{ for } j\neq k}} S(x_{i_1},x_{i_1},x_{i_2},x_{i_3},x_{i_4},x_{i_5})$$

$$+ \sum_{\substack{i_1,i_2,i_3,i_4,i_5,i_6\in[n] \\ i_j\neq i_k \text{ for } j\neq k}} S(x_{i_1},x_{i_2},x_{i_3},x_{i_4},x_{i_5},x_{i_6})\Big) \ . \tag{9}$$

Observing (8) and (9), we conclude that the proof of (2) reduces to the proof of the following inequality:

$$T(x_1,...,x_6) \leq \frac{4}{3} S(x_1,...,x_6)\ ,\ x_1,...,x_6 \in \mathbb{R}^d \ . \tag{10}$$

To establish (10), some lemmas from three-dimensional affine geometry are required.

Lemma 3.2. Let $x_1,...,x_6$ be vectors in $\mathbb{R}^3$, where $x_4,\ x_5,\ x_6$ are linearly dependent. The following formulas are true:
i) $(x_1 \wedge x_2) \wedge (x_2 \wedge x_3) = det(x_1,\ x_2,\ x_3) \cdot x_2$.
ii) $det(x_1 \wedge x_2\ ,\ x_3 \wedge x_4\ ,\ x_5 \wedge x_6) = det(x_1,x_2,x_4) \cdot det(x_3,x_5,x_6)$.

Assertion (i) is a well known property of the vector product in $\mathbb{R}^3$. To prove (ii), we may assume that there exist numbers $\lambda_4,\ \lambda_6$ such that $x_5 = \lambda_4 x_4 + \lambda_6 x_6$. Then, by (i) we have:

$$det(x_1 \wedge x_2\ ,\ x_3 \wedge x_4\ ,\ x_5 \wedge x_6) = \lambda_4 < x_1 \wedge x_2\ ,\ (x_3 \wedge x_4) \wedge (x_4 \wedge x_6) >$$

$$= \lambda_4 \cdot det(x_1,x_2,x_4) \cdot det(x_3,x_4,x_6) = det(x_1,x_2,x_4) \cdot det(x_3,\lambda_4 x_4 + \lambda_6 x_6, x_6)$$

$$= det(x_1,x_2,x_4) \cdot det(x_3,x_5,x_6) \ . \quad \square$$

Lemma 3.3. Let $x_1, ..., x_6$ be vectors in $\mathbb{R}^3$. If two of them are parallel, then

$$T(x_1, ..., x_6) = \frac{4}{3} S(x_1, ..., x_6) \ .$$

Proof. Obviously, we may assume that all $x_1, ..., x_6$ are unit vectors. Also, by symmetry, one can take $x_5 = x_6$. It follows by the previous lemma that

$$\mid det(x_5 \wedge x_{i_1}, \ x_5 \wedge x_{i_2}, \ x_{i_3} \wedge x_{i_4}) \mid = \mid det(x_5, x_{i_1}, x_{i_2}) \cdot det(x_5, x_{i_3}, x_{i_4}) \mid$$

$$= \mid det(x_5 \wedge x_{i_3}, \ x_5 \wedge x_{i_4}, \ x_{i_1} \wedge x_{i_2}) \mid \ ,$$

where $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Thus,

$$T(x_1, ..., x_6) = 2 \cdot 2^3 \cdot 3! \sum_{\substack{i_1 < i_2, \ i_3 < i_4 \\ \{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}}} \mid det(x_5 \wedge x_{i_1}, \ x_5 \wedge x_{i_2}, \ x_{i_3} \wedge x_{i_4}) \mid$$

$$= 2 \cdot 2^3 \cdot 3! \sum_{\substack{i_1 < i_2, \ i_3 < i_4 \\ \{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}}} \mid det(x_5, x_{i_1}, x_{i_2}) \cdot det(x_5, x_{i_3}, x_{i_4}) \mid$$

$$= \frac{2 \cdot 2^3 \cdot 3!}{2 \cdot 2} \sum_{\{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}} \mid det(x_5, x_{i_1}, x_{i_2}) \cdot det(x_5, x_{i_3}, x_{i_4}) \mid$$

$$= \frac{2 \cdot 2^3 \cdot 3!}{2^2 \cdot 2 \cdot 3^2} S(x_1, ..., x_6) = \frac{4}{3} S(x_1, ..., x_6) \ . \quad \square$$

Let Z be a zonotope in $\mathbb{R}^3$, which is the sum of five line segments. As mentioned above, one can write $Z = \sum_{i=1}^{6} [-x_i, x_i]$, for some $x_1, ..., x_6$ in $\mathbb{R}^3$ with $x_5 = x_6$. The previous lemma combined with (8) and (9) ensures that $|\Pi Z| = 8|Z|^2$.

Suppose, now, that $x_1, ..., x_s$ are vectors in $\mathbb{R}^3$. We set for simplicity $\mathcal{E}(x_1, ..., x_s)$ to be the set of all planes through 0, spanned be pairs of vectors from $x_1, ..., x_s$.

Lemma 3.4. Let $x_1, ..., x_6$ be vectors that span $\mathbb{R}^3$, such that any two of them are not parallel. Assume that for every $i = 1, ..., 6$, there exist two different planes $E_1, \ E_2$ from $\mathcal{E}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_6)$, that contain x_i. Then, after a possible rearrangement of indices, the sets of coplanar vectors from $x_1, ..., x_6$ are exactly the following:

$$\{x_1, x_2, x_3\} \ , \ \{x_2, x_4, x_5\} \ , \ \{x_1, x_5, x_6\} \ , \ \{x_3, x_4, x_6\} \ .$$

Proof. Clearly, any five vectors from $x_1, ..., x_6$ cannot lie in the same plane. We assume, without loss of generality that x_1, x_2, x_3 are linearly dependent. Since there exists a plane E in $\mathcal{E}(x_1, x_3, ..., x_6)$ that contains x_2, different than the one spanned by x_1, x_3, we may assume that x_2, x_4, x_5 are linearly dependent, while each one of x_4, x_5 is not coplanar with x_1, x_3.

Similarly, either x_1 is linearly dependent with x_5, x_6 or it is linearly dependent with x_4, x_6. We may assume that the first case occurs. Now, x_6 cannot be contained in any of the planes spanned by x_4, x_5 and x_1, x_2 (in the opposite case, five vectors from the x_i's would be coplanar). This forces x_3 to be coplanar with x_4, x_6.

We have shown that these sets are indeed linearly dependent. If there existed another subset of $\{x_1, ..., x_6\}$ with this property, five vectors from the x_i's would be coplanar, which is impossible. □

The key to the proof of (10) will be the next lemma.

Lemma 3.5. Let $x_1, ..., x_6$ be vectors, for which the conclusion of Lemma 3.4 holds. Then

$$T(x_1, ..., x_6) < \frac{4}{3} S(x_1, ..., x_6) \ .$$

Proof. Consider the following subsets of the set U of summands in $T(x_1, ..., x_6)$:

$$U_{ij} = \Big\{ \mid det(x_i \wedge x_j \ , \ x_{i_1} \wedge x_{i_2} \ , \ x_{i_3} \wedge x_{i_4}) \mid \neq 0 \ \Big| \ \{i_1, i_2, i_3, i_4\} = \{1, ..., 6\} \backslash \{i, j\} \Big\},$$

$i, j = 1, ..., 6, \ i \neq j$. It is clear, that the sets U_{12}, U_{23}, U_{13}, U_{24}, U_{25}, U_{45}, U_{26} cover U. It follows from Lemma 3.2 that the elements of U_{12} are exactly the terms of the form:

$$\mid det(x_1, x_2, x_{i_1}) \cdot det(x_{i_2}, x_{i_3}, x_{i_4}) \mid \neq 0 \ , \ \{i_1, i_2, i_3, i_4\} = \{3, 4, 5, 6\} \ .$$

Similar expressions can be derived for the elements of U_{23} and U_{13}. Hence, since $\mid det(x_1, x_2, x_3) \mid = 0$ and U_{12}, U_{23}, U_{13} are disjoint the sum of all terms that belong to $V_1 := U_{12} \cup U_{23} \cup U_{13}$, is a constant multiple of $S(x_1, ..., x_6)$. One may easily compute this constant to be $2/3$.

Similarly, the sum of all terms contained in $V_2 := U_{24} \cup U_{25} \cup U_{45}$ also equals $2/3 \cdot S(x_1, ..., x_6)$.

Clearly, terms of the form (each one counted $2^3 \cdot 3!$-times)
$\mid det(x_4 \wedge x_5 \ , \ x_1 \wedge x_6 \ , \ x_2 \wedge x_3) \mid$, $\mid det(x_4 \wedge x_5 \ , \ x_1 \wedge x_2 \ , \ x_3 \wedge x_6) \mid$, $\mid det(x_2 \wedge x_5 \ , \ x_1 \wedge x_3 \ , \ x_4 \wedge x_6) \mid$ belong both to V_1 and V_2. Thus, if A is the sum of terms from $V_1 \cup V_2$, we have:

$$A \leq \frac{4}{3} S(x_1, ..., x_6) - 2^3 \cdot 3! \Big[\mid det(x_4 \wedge x_5 \ , \ x_1 \wedge x_6 \ , \ x_2 \wedge x_3) \mid$$

$$+\mid det(x_4 \wedge x_5\ ,\ x_1 \wedge x_2\ ,\ x_3 \wedge x_6) \mid + \mid det(x_2 \wedge x_5\ ,\ x_1 \wedge x_3\ ,\ x_4 \wedge x_6) \mid\Big]$$

$$= \frac{4}{3}S(x_1, ..., x_6) - 2^3 \cdot 3!\Big[\mid det(x_4, x_5, x_3) \cdot det(x_1, x_2, x_6) \mid$$

$$+\mid det(x_4, x_5, x_1) \cdot det(x_2, x_3, x_6) \mid + \mid det(x_1, x_3, x_5) \cdot det(x_2, x_4, x_6) \mid\Big]\ ,$$

where we used once again Lemma 3.2.

Next, we observe that x_1, x_5, x_6 are coplanar, so by Lemma 3.2 we have:

$$\mid det(x_2 \wedge x_6\ ,\ x_1 \wedge x_5\ ,\ x_3 \wedge x_4) \mid = \mid det(x_1, x_2, x_5) \cdot det(x_3, x_4, x_6) \mid = 0\ .$$

Consequently,

$$U_{26} \setminus (V_1 \cup V_2) = \Big\{\mid det(x_2 \wedge x_6\ ,\ x_1 \wedge x_4\ ,\ x_3 \wedge x_5) \mid\Big\}$$

and since $\mid det(x_4, x_5, x_3) \cdot det(x_1, x_2, x_6) \mid > 0$, we conclude:

$$T(x_1, ..., x_6) < \frac{4}{3}S(x_1, ..., x_6) + 2^3 \cdot 3!\Big[\mid det(x_2 \wedge x_6\ ,\ x_1 \wedge x_4\ ,\ x_3 \wedge x_5) \mid$$

$$-\mid det(x_4, x_5, x_1) \cdot det(x_2, x_3, x_6) \mid - \mid det(x_1, x_3, x_5) \cdot det(x_2, x_4, x_6) \mid\Big]\ .$$

Finally, by assumption, there exist numbers $\lambda_4,\ \lambda_6$ such that $x_3 = \lambda_4 x_4 + \lambda_6 x_6$. Using the fact that x_1, x_5, x_6 are coplanar, we have:

$$\mid det(x_2 \wedge x_6\ ,\ x_1 \wedge x_4\ ,\ x_3 \wedge x_5) \mid \leq \mid \lambda_4 \cdot det(x_2 \wedge x_6\ ,\ x_1 \wedge x_4\ ,\ x_4 \wedge x_5) \mid$$

$$+\mid \lambda_6 \cdot det(x_2 \wedge x_6\ ,\ x_1 \wedge x_4\ ,\ x_6 \wedge x_5) \mid$$

$$= \mid \lambda_4 \cdot det(x_4, x_5, x_1) \cdot det(x_4, x_2, x_6)| + \mid \lambda_6 \cdot det(x_4, x_5, x_6) \cdot det(x_1, x_2, x_6)|$$

$$= \mid det(x_3, x_5, x_1) \cdot det(x_4, x_2, x_6) \mid + \mid det(x_4, x_5, x_1) \cdot det(x_2, x_3, x_6) \mid\ ,$$

completing the proof. $\square$

Proof of (10):

If the assumptions of Lemma 3.3 or Lemma 3.4 are true or $S(x_1, ..., x_6) = 0$, the assertion is obvious. In any other case, there exists some i in $\{1, ..., 6\}$, such that x_i belongs to at most one plane from $\mathcal{E}(x_1, ...x_{i-1}, x_{i+1}, ..., x_6)$. It is easy, then, to see that there exist real numbers $t_1 < 0 < t_2$ and a vector ν such that: For all $E \in \mathcal{E}(x_1, ...x_{i-1}, x_{i+1}, ..., x_6)$, we have:

$$x_i \in E\ \Leftrightarrow\ x_i + t\nu \in E\ ,\ \text{for all}\ \ t \in (t_1, t_2)$$

and also

$$\#\mathcal{E}(x_1,...x_{i-1},x_i+t_j\nu,x_{i+1},...,x_6)<\#\mathcal{E}(x_1,...,x_6)\ ,\ j=1,2. \tag{11}$$

We note here that this fact will be also used in the next section.

Consequently, the function

$$[t_1,t_2]\ni t\mapsto S(x_1,...x_{i-1},x_i+t\nu,x_{i+1},...,x_6)$$

is affine. Thus, by the convexity of $T(x_1,...x_{i-1},x_i+t\nu,x_{i+1},...,x_6)$ and by Lemma 3.1, we conclude that

$$\frac{T}{S}(x_1,...,x_6)\le\frac{T}{S}(x_1,...,x_{i-1},x_i+t_j\nu,x_{i+1},...,x_6)\ ,$$

for $j=1$ or 2 (clearly, $S(x_1,...,x_i+t_j\nu,...,x_6)$ cannot be zero for both $j=1,2$; in the opposite case, $S(x_1,...,x_6)$ would be zero).

We may repeat this procedure as many times as needed. Nevertheless, as (11) shows, after only a finite number of steps we will have found vectors $z_1,...,z_6$ for which the conditions of Lemma 3.3 or Lemma 3.4 are true and also $(T/S)(x_1,...,x_6)\le(T/S)(z_1,...,z_6)$. $\square$

4. Characterization of extremal zonoids

Lemma 4.1. Let $x_1,...,x_6$ be vectors in $\mathbb{R}^3$. If four of them are coplanar, then

$$T(x_1,...,x_6)=\frac{4}{3}S(x_1,...,x_6)\ .$$

Proof. Suppose e.g. that x_1,x_2,x_3,x_4 are coplanar. If (i_1,i_2,i_3,i_4) is a permutation on $\{1,2,3,4\}$, it is clear that

$$|\ det(x_{i_1}\wedge x_{i_2}\ ,\ x_{i_3}\wedge x_{i_4}\ ,\ x_{i_5}\wedge x_{i_6})\ |=0=|\ det(x_{i_1},x_{i_2},x_{i_3})\cdot det(x_{i_4},x_{i_5},x_{i_6})\ |\ .$$

Also, by Lemma 3.2, it follows that

$$|\ det(x_{i_1}\wedge x_{i_2}\ ,\ x_{i_3}\wedge x_5\ ,\ x_{i_4}\wedge x_6)\ |=|\ det(x_{i_1},x_{i_2},x_5)\cdot det(x_{i_3},x_{i_4},x_6)\ |$$

$$=|\ det(x_{i_1}\wedge x_{i_2}\ ,\ x_{i_4}\wedge x_5\ ,\ x_{i_3}\wedge x_6)\ |\ .$$

Hence,

$$T(x_1,...,x_6)=2^3\cdot 3!\sum_{\substack{(i_1,i_2,i_3,i_4)\in S_4\\ i_1<i_2}}|\ det(x_{i_1}\wedge x_{i_2}\ ,\ x_{i_3}\wedge x_5\ ,\ x_{i_4}\wedge x_6)\ |$$

$$= 2 \cdot 2^3 \cdot 3! \sum_{\substack{(i_1,i_2,i_3,i_4)\in S_4 \\ i_1<i_2, i_3<i_4}} \mid det(x_{i_1}, x_{i_2}, x_5) \cdot det(x_{i_3}, x_{i_4}, x_6) \mid = \frac{4}{3} S(x_1, ..., x_6) ,$$

where S_4 is the set of permutations on $\{1, 2, 3, 4\}$. $\square$

Suppose, now, that Z is the sum of at least six line segments $[-x_i, x_i]$, $i = 1, ..., n$ (as mentioned above, this is no loss of generality). If Z is the sum of a cylinder and a line segment, then for any six vectors from $x_1, ..., x_n$, at least two are collinear or at least four are coplanar. The previous Lemma, Lemma 3.3, (8) and (9) show that $|\Pi Z| = 2^3|Z|^2$. Since the same is true when Z is the sum of five line segments, we only have to prove the "only if" part in Theorem 1.

It is now clear that the problem of characterization of zonoids, for which equality in (2) is attained, reduces to the determination of the 6-tuples $(x_1, ..., x_6)$ such that $T(x_1, ..., x_6) = (4/3) \cdot S(x_1, ..., x_6)$. If the conditions of Lemma 3.3 or Lemma 4.1 hold, the last equality is true. In what follows, we will show that these are the only possible equality cases.

To accomplish this, we need a series of geometric lemmas. The proof of the following is obvious.

Lemma 4.2. Let E be a plane in $\mathbb{R}^3$, that does not contain 0 and $y_1, ..., y_5$ be points in E, with $y_4 \neq y_5$. Suppose, also, that y_3 lies in the line segment $[y_1, y_2]$ and y_i is not collinear with y_1, y_2, $i = 4, 5$. If x_i is the position vector of y_i, $i = 1, ..., 5$, then there exists a vector ν in $\mathbb{R}^3$ and real numbers $t_1 < 0 < t_2$, such that $x_3 + t_i\nu$ is parallel to x_i, $i = 1, 2$ and $det(x_3 + t\nu, x_4, x_5) \neq 0$ for all t in (t_1, t_2), if and only if y_3 is an interior point of $[y_1, y_2]$, while at the same time the line $aff\{y_4, y_5\}$ and the interior of the segment $[y_1, y_2]$ are disjoint.

In order to make use of the previous lemma, we observe that the ratio $T/S(x_1, ..., x_6)$ is independent of the length and the orientation of the x_i's. Thus, we may assume that the endpoints of $x_1, ..., x_6$ all lie in the same plane in $\mathbb{R}^3$, not containing the origin.

Let $A = \{x_1, ..., x_6\}$ be a set of six vectors in $\mathbb{R}^3$. We say that A has the (N)-property, if no two of $x_1, ..., x_6$ are parallel and no four of them are coplanar.

Lemma 4.3. Let $\{x_1, ..., x_6\}$ be a set of vectors in $\mathbb{R}^3$ having the (N)-property, where x_i is the position vector of some point y_i, $i = 1, ..., 6$. We assume the following:

i) $y_1, ..., y_6$ are coplanar and y_6 is an interior point of the segment $[y_1, y_2]$.

ii) For every i, j, $i \neq j$, in $\{3, 4, 5\}$ the line spanned by the points y_i, y_j, and

the interior of $[y_1, y_2]$ are disjoint.
iii) There exists some permutation (k_1, k_2, k_3) of $\{3, 4, 5\}$ and some interior point y of $[y_1, y_2]$, such that the lines $aff\{y_1, y_{k_1}\}$, $aff\{y_2, y_{k_2}\}$, $aff\{y, y_{k_3}\}$ are either parallel or they have a common point, which is different from y_3, y_4, y_5.
Then,

$$T(x_1, ..., x_6) < \frac{4}{3} S(x_1, ..., x_6) \ .$$

Proof. Let us assume that $T(x_1, ..., x_6) = (4/3) \cdot S(x_1, ..., x_6)$. According to the previous lemma, there exist numbers $t_1 < 0 < t_2$ and a vector ν, such that $x_6 + t_i\nu$ is parallel to x_i, $i = 1, 2$, and the quantity $S(x_1, ..., x_5, x_6 + t\nu)$ is affine (and positive) in $[t_1, t_2]$, as a function of t. This, combined with (10), Lemma 3.1 and the fact that $T(x_1, ..., x_5, x_6 + t\nu)$ is convex on t in $[t_1, t_2]$, shows immediately that the function

$$[t_1, t_2] \ni t \mapsto \frac{T}{S}(x_1, ..., x_5, x_6 + t\nu)$$

is constant. In particular, $T(x_1, ..., x_5, x_6 + t\nu)$ must be affine in $[t_1, t_2]$.

Since it is clear that there does not exist a $j \in \{3, 4, 5\}$ such that y_j is collinear with y_1, y_2 (in the opposite case, x_1, x_2, x_6, x_j would be coplanar), the point y of assumption (iii) is unique. In other words, for some permutation (k_1, k_2, k_3) of $\{3, 4, 5\}$, the lines $\varepsilon_1 := aff\{y_1, y_{k_1}\}$, $\varepsilon_2 := aff\{y_2, y_{k_2}\}$, $aff\{y, y_{k_3}\}$ are parallel or have a common point, while for each y' in the interior of $[y_1, y_2]$, different from y, the lines ε_1, ε_2, $aff\{y', y_{k_3}\}$ neither are parallel nor contain a common point.

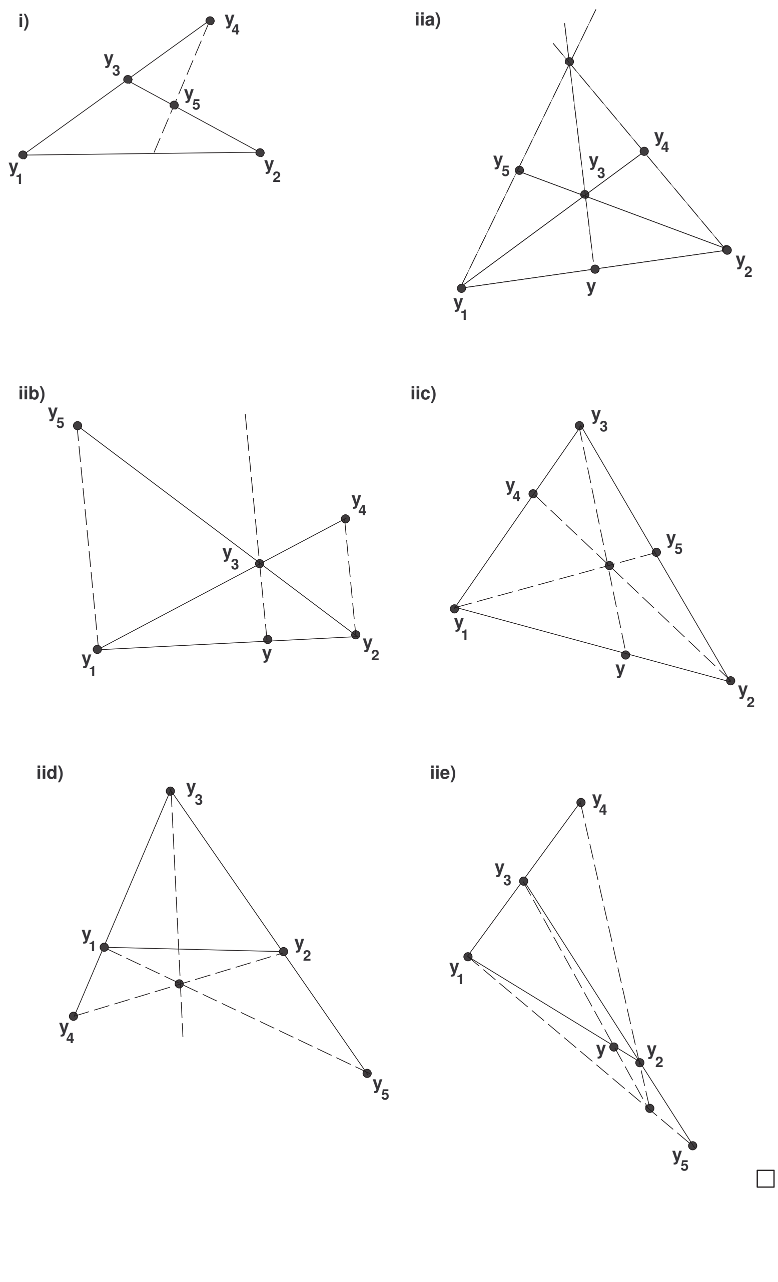
Consequently, for $t_1 < t < t_2$, the intersection of the planes $span\{x_1, x_{k_1}\}$, $span\{x_2, x_{k_2}\}$, $span\{x_6 + t\nu, x_{k_3}\}$ is non-trivial if and only if $x_6 + t\nu$ is parallel to the position vector of y. This shows that the quantity

$$|\ det(x_1 \wedge x_{k_1}\ ,\ x_2 \wedge x_{k_2}\ ,\ (x_6 + t\nu) \wedge x_{k_3})\ |$$

is zero for a unique interior point t of $[t_1, t_2]$.

It follows that the function $T(x_1, ..., x_5, x_6 + t\nu)$ cannot be affine in $[t_1, t_2]$, proving our claim. □

Lemma 4.4. Let $y_1, ..., y_5$ be points lying in the same plane, such that y_1, y_3, y_4 are collinear, y_2, y_3, y_5 are collinear and there are no other sets of three collinear points among them. Exactly one of the following are true:
i) $aff\{y_4, y_5\} \cap int[y_1, y_2] \neq \emptyset$.
ii) The assumption (iii) in Lemma 4.3 holds true.
Proof. Up to a possible rearrangement of indices, there exist exactly the following cases:


i)
y1
y2
y3
y4
y5
iia)
y1
y2
y3
y4
y5
y
iib)
y1
y2
y3
y4
y5
y
iic)
y1
y2
y3
y4
y5
y
iid)
y1
y2
y3
y4
y5
iie)
y1
y2
y3
y4
y5
y

□

Lemma 4.5. Let $\{y_1, ..., y_5\}$ be discrete points, lying in the same plane, such that:
i) y_5 is an interior point of the line segment $[y_3, y_4]$.
ii) Each one of the segments $[y_1, y_2]$, $[y_3, y_4]$ is contained in one of the two open half-planes defined by the other one.
Then, for some choice of k_1, k_2, there exists an interior point y of $[y_1, y_2]$, such that the lines $aff\{y, y_5\}$, $aff\{y_1, y_{k_1}\}$, $aff\{y_2, y_{k_2}\}$ have a common point, where $\{k_1, k_2\} = \{3, 4\}$.

Proof. The vertices of the polygon $P = conv\{y_1, y_2, y_3, y_4\}$ are exactly the points y_1, y_2, y_3, y_4. Therefore, the line defined by y_5 and the intersection point of the diagonals of P, crosses $[y_1, y_2]$ at one of its interior points. □

Lemma 4.6. Suppose that the set $\{x_1, ..., x_6\}$ satisfies the (N)-property. Assume, furthermore, that there is no plane in $\mathcal{E}(x_1, ..., x_5)$, that contains x_6. Then, there exist a vector ν and real numbers $t_1 < 0 < t_2$ with the following properties:
i) There exists a plane E_i from $\mathcal{E}(x_1, ..., x_5)$ that contains $x_6 + t_i\nu$, $i = 1, 2$.
ii) For all t in (t_1, t_2), there is no plane from $\mathcal{E}(x_1, ..., x_5)$, that contains $x_6 + t\nu$.
iii) The set $\{x_1, ..., x_5, x_6 + t_i\nu\}$ satisfies the (N)-property, for $i = 1$ or 2.

Proof. If there exists at most one 3-tuple of coplanar vectors from $x_1, ..., x_6$, our claim follows easily. If there exist at least two such 3-tuples, let G be the open convex angle defined by the corresponding planes, so that G contains x_6. Clearly, there exists some plane E, that contains exactly two vectors from $x_1, ..., x_5$ but not x_6, the intersection of E with G is not empty and for some point x in E the interior of the segment $[x, x_6]$ and any plane from $\mathcal{E}(x_1, ..., x_5)$ are disjoint. The result follows. □

The proof of the following fact is easy and will be omitted.

Lemma 4.7. Suppose that the set $\{x_1, ..., x_6\}$ has the (N)-property and for every $i = 1, .., 6$, there exists a plane E_i from $\mathcal{E}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_6)$ that contains x_i. Up to a possible rearrangement of indices, one of the following is true:
i) x_1, x_2, x_3 are coplanar and x_4, x_5, x_6 are coplanar.
ii) x_1, x_2, x_3 are coplanar, x_3, x_4, x_5 are coplanar and x_1, x_5, x_6 are coplanar.

Now we are ready to prove the key fact mentioned at the beginning of this section.

Lemma 4.8. The set $\{x_1, ..., x_6\}$ satisfies the (N)-property, if and only if

$$T(x_1, ..., x_6) < \frac{4}{3} S(x_1, ..., x_6) \ .$$

Proof. It suffices to prove the "only if" part. Suppose that the set $\{x_1, ..., x_6\}$ satisfies the (N)-property. If the assumptions of Lemma 3.4 are true, the assertion is true by Lemma 3.5.

Case I: Assume that the following are true:
a) For every $i = 1, ..., 6$, there exists some plane E_i from $\mathcal{E}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_6)$ that contains x_i.
b) There exists an i, $i = 1, ..., 6$, a vector ν and real numbers $t_1 < 0 < t_2$, so that for all $k, l = 1, ..., 6$ and for all t in (t_1, t_2), $det(x_i + t\nu, x_k, x_l) = 0$, if and only if $det(x_i, x_k, x_l) = 0$ and, furthermore, the sets $\{x_1, ..., x_{i-1}, x_i + t_j\nu, x_{i+1}, ..., x_6\}$ do not satisfy the (N)-property, $j = 1, 2$.

By assumption (a) and Lemma 4.7, there are two possibilities (rearranging the indices, if necessary):
i) x_1, x_2, x_3 are contained in some plane E_1 and x_4, x_5, x_6 are contained in some other plane E_2. Since the (N)-property holds, replacing x_i with $-x_i$ if necessary, we may assume that x_1, x_2, x_3 are contained in the same open half-space of E_2 and x_4, x_5, x_6 are contained in the same open half-space of E_1. One can check that we can, simultaneously, take x_i to be the position vector of some point y_i, $i = 1, ..., 6$, where $y_1, ..., y_6$ are coplanar. Then, it is clear that five of the points $y_1, ..., y_6$ satisfy the assumptions of Lemma 4.5, thus by Lemma 4.3 we obtain $T(x_1, ..., x_6) < (4/3) \cdot S(x_1, ..., x_6)$.
ii) $\{x_1, x_2, x_3\}$, $\{x_2, x_4, x_5\}$, $\{x_1, x_5, x_6\}$ are sets of linearly dependent vectors. Then, we may assume that there exists some vector ν and real numbers $t_1 < 0 < t_2$, so that for all t in (t_1, t_2), we have $det(x_3 + t\nu, x_k, x_l) \neq 0$ if and only if $det(x_3, x_k, x_l) \neq 0$, while the sets $\{x_1, x_2, x_3 + t_j\nu, x_4, x_5, x_6\}$, $j = 1, 2$, do not satisfy the (N)-property. If for $j = 1$ or 2, four of the vectors x_1, x_2, $x_3 + t_j\nu$, x_4, x_5, x_6 were coplanar, then x_4, x_5, x_6 would also be coplanar and $x_1, ..., x_6$ would not span $\mathbb{R}^3$. This forces $x_3 + t_j\nu$ to be parallel to x_j, $j = 1, 2$. As before, we may assume that the x_i's are the position vectors of some coplanar points $y_1, ..., y_6$ respectively. It is clear that we can apply Lemma 4.4 for the points y_1, y_2, y_4, y_5, y_6. If (i) of Lemma 4.4 is satisfied, then Lemma 4.2 contradicts to our assumption in the present Lemma. Therefore, the assertion (ii) of Lemma 4.4 holds, hence $T(x_1, ..., x_6) < (4/3) \cdot S(x_1, ..., x_6)$.

By Lemma 4.6, the only remaining case is the following:

Case II: There exist an index i from $\{1, ..., 6\}$, a vector ν and an interval $[t_1, t_2]$ that contains 0 in its interior, which is maximal under the assumption that the following are true:
a) For all t in (t_1, t_2) and for all $k, l = 1, ..., 6$, $k, l \neq i$, $x_i + t\nu$, x_k, x_l are

coplanar, if and only if x_i, x_k, x_l are coplanar.
b) The set $\{x_1, ..., x_{i-1}, x_i + t_j\nu, x_{i+1}, ..., x_6\}$ satisfies the (N)-property, for $j = 1$ or 2.

Let us assume that equality holds in (10). By assumption (a), it follows that the function

$$[t_1, t_2] \ni t \mapsto S(x_1, ...x_{i-1}, x_i + t\nu, x_{i+1}, ..., x_6) \ ,$$

is affine, thus Lemma 3.1, combined with (10), implies that

$$\frac{T}{S}(x_1, ...x_{i-1}, x_i + t\nu, x_{i+1}, ..., x_6) = \frac{4}{3} \ , \quad t \in [t_1, t_2] \ .$$

Hence, by assumption, it is clear that there exists a set of vectors $\{z_1, ..., z_6\}$ with the (N)-property, such that $T(z_1, .., z_6) = (4/3) \cdot S(z_1, ..., z_6)$ and $\#\mathcal{E}(z_1, ..., z_6) < \#\mathcal{E}(x_1, ..., x_6)$. Clearly, after a finite number of repetitions of the same procedure, we will have constructed a set of six vectors with the (N)-property, that satisfies the assumptions of Lemma 3.4 or falls into Case I. This is impossible and the conclusion follows. $\Box$

The proof of the remaining part of Theorem 1 follows easily from Lemma 4.8. Indeed, let Z be a zonotope in $\mathbb{R}^3$ with support function

$$h_Z(x) = \int_{S^2} | < x, y > | \ d\mu(y) \ ,$$

where $\mu(\cdot) = \sum_{i=1}^{n} \alpha_i \delta_{x_i}(\cdot)$ and $\delta_{x_i}(\cdot)$ is the Dirac measure in x_i, for some unit vectors x_i and some positive numbers α_i ($n \geq 6$). By Lemma 3.3, (8) and (9), we have

$$6!\left(\frac{2^6}{3!}\right)^{-1}\left[2^3|Z|^2 - |\Pi Z|\right] = 6! \sum_{\substack{i_1,...,i_6 \in [n] \\ i_1 < ... < i_6}} \alpha_{i_1} ... \alpha_{i_6} \left[\frac{4}{3} S(x_{i_1}, ..., x_{i_6}) - T(x_{i_1}, ..., x_{i_6})\right]$$

$$= \sum_{i_1,...,i_6 \in [n]} \alpha_{i_1} ... \alpha_{i_6} \left[\frac{4}{3} S(x_{i_1}, ..., x_{i_6}) - T(x_{i_1}, ..., x_{i_6})\right] .$$

or

$$6!\left(\frac{2^6}{3!}\right)^{-1}\left[2^3|Z|^2 - |\Pi Z|\right] = \int_{x_1 \in S^2} ... \int_{x_6 \in S^2} \varphi(x_1, ..., x_6) \ d\mu(x_1) ... d\mu(x_6) \ ,$$

where we set $\varphi := (4/3) \cdot S - T$. By approximation, the last identity holds for any measure on S^2, thus for every three-dimensional zonoid.

Now, if Z is not the sum of five line segments or the sum of a cylinder and a line segment, there clearly exists a set of vectors $\{y_1, ..., y_6\}$, contained in the support of μ, that satisfies the (N)-property. By the continuity of φ and the fact that $\varphi(y_1, ..., y_6) > 0$, we have

$$\int_{x_1 \in S^2} \cdots \int_{x_6 \in S^2} \varphi(x_1, ..., x_6)\ d\mu(x_1)...d\mu(x_6) > 0\ ,$$

which completes the proof. $\square$

5. Cones and double cones

Suppose that $K = conv(P \cup \{e_3\})$ is a cone in $\mathbb{R}^3$, where P is a convex body in $\mathbb{R}^2 \times \{0\}$. For our purpose, we may assume that P is a polygon that contains the origin. Let $A_1, ..., A_n$ be the edges of P. Set, also, h_i to be the outer normal vector to A_i, of length equal to the distance of A_i from the origin and take vectors a_i, parallel to A_i, which have length equal to the length of A_i, $i = 1, ..., n$. We may choose the orientations of the a_i's, so that $det(a_i, h_i) > 0$, $i = 1, ..., n$.

Now the facets of K are exactly the sets

$$P\ ,\ F_i := conv(A_i \cup \{e_3\})\ ,\ i = 1, ..., n\ .$$

Let x_i be the outer unit normals to F_i, $i = 1, ..., n$. Then, since $-e_3$ is the outer unit normal vector to P, by (6), we have:

$$|\Pi K| = \sum_{\{i_1,i_2,i_3\} \subseteq [n]} |F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\ det(x_{i_1}, x_{i_2}, x_{i_3})\ | + \sum_{\{i_1,i_2\} \subseteq [n]} |P| \cdot |F_{i_1}| \cdot |F_{i_2}| \cdot |\ det(x_{i_1}, x_{i_2}, e_3)\ | \tag{12}$$

A crucial observation for what follows is the fact that all terms of the form $|F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\ det(x_{i_1}, x_{i_2}, x_{i_3})\ |$ are non-zero. One can easily see this by taking a suitable affine transformation that maps F_{i_1}, F_{i_2} to facets that are parallel to the vector e_3. Then, for any i_3 different than i_1, i_2, the image of F_{i_3} through this transformation is necessarily not parallel to e_3.

Clearly, the vector $h_i - e_3$ is parallel to the facet F_i. Thus, the vector $a_i \wedge (h_i - e_3)$ is orthogonal to F_i, hence a multiple of x_i. Moreover, since $h_i - e_3$ is orthogonal to a_i,

$$|a_i \wedge (h_i - e_3)| = |a_i| \cdot |h_i - e_3| = 2|F_i|\ ,\ i = 1, ..., n\ .$$

Consequently, we have shown that

$$|F_i| \cdot x_i = \pm \frac{1}{2} a_i \wedge (h_i - e_3) \tag{13}$$

We may now use (13), Lemma 3.2 (i) and the fact that the vectors $a_1 \wedge h_1$, $a_2 \wedge h_2$ are parallel, to deal with every term of (12) separately:

$$|P| \cdot |F_1| \cdot |F_2| \cdot |\ det(x_1, x_2, e_3)\ | = |P| \cdot \frac{1}{4} |\ det(a_1 \wedge (h_1 - e_3), a_2 \wedge (h_2 - e_3), e_3)\ |$$

$$= \frac{1}{4} |P| \cdot |\ det(a_1 \wedge e_3, a_2 \wedge e_3, e_3)\ | = \frac{1}{4} |P| \cdot |\ det_{2\times 2}(a_1, a_2)\ | \ .$$

Thus,

$$\sum_{\{i_1, i_2\} \subseteq [n]} |P| \cdot |F_{i_1}| \cdot |F_{i_2}| \cdot |\ det(x_{i_1}, x_{i_2}, e_3)\ | = \frac{1}{4} |P| \cdot \sum_{\{i_1, i_2\} \subseteq [n]} |\ det(a_{i_1}, a_{i_2})\ |$$

$$= \frac{1}{4} |P| \cdot |\Pi P| \ . \tag{14}$$

Moreover,

$$\left|\ det\Big(a_1 \wedge (h_1 - e_3), a_2 \wedge (h_2 - e_3), a_3 \wedge (h_3 - e_3)\Big)\ \right|$$

$$= \Big|\ det(-a_1 \wedge e_3, -a_2 \wedge e_3, a_3 \wedge h_3) + det(-a_1 \wedge e_3, a_2 \wedge h_2, -a_3 \wedge e_3)$$

$$+ det(a_1 \wedge h_1, -a_2 \wedge e_3, -a_3 \wedge e_3)\ \Big|$$

$$= |\ < (-a_1 \wedge e_3) \wedge (e_3 \wedge a_2), a_3 \wedge h_3 > + < (-a_1 \wedge e_3) \wedge (e_3 \wedge -(a_3)), a_2 \wedge h_2 >$$

$$+ < a_1 \wedge h_1, (-a_2 \wedge e_3) \wedge (e_3 \wedge a_3) >\ |$$

$$= |\ det(-a_1, e_3, a_2) \cdot det(e_3, a_3, h_3) + det(-a_1, e_3, -a_3) \cdot det(e_3, a_2, h_2)$$

$$+ det(-a_2, e_3, a_3) \cdot det(a_1, h_1, e_3)\ |$$

$$= |\ det_{2\times 2}(a_1, a_2) \cdot det_{2\times 2}(a_3, h_3) + det_{2\times 2}(a_3, a_1) \cdot det_{2\times 2}(a_2, h_2)$$

$$+ det_{2\times 2}(a_2, a_3) \cdot det_{2\times 2}(a_1, h_1)\ | \ .$$

Since $|\ conv(\{0\} \cup A_i)\ | = det(a_i, h_i)/2$, it follows that

$$|F_1| \cdot |F_2| \cdot |F_3| \cdot |det(x_1, x_2, x_3)| = \frac{1}{4} \cdot \Big|\ det(a_2, a_3) \cdot |\ conv(\{0\} \cup A_1)\ |$$

$$+ det(a_3, a_1) \cdot |\ conv(\{0\} \cup A_2)\ | + det(a_1, a_2) \cdot |\ conv(\{0\} \cup A_3)\ |\ \Big| \ . \tag{15}$$

If we assume, in addition, that A_1, A_2 are adjacent edges, then

$$\mid conv(\{0\} \cup A_1 \cup A_2) \mid = \mid conv(\{0\} \cup A_1) \mid + \mid conv(\{0\} \cup A_2) \mid .$$

In this case,

$$det(a_3, a_1) \cdot \mid conv(\{0\} \cup A_2) \mid + det(a_2, a_3) \cdot \mid conv(\{0\} \cup A_1) \mid$$

$$= det(a_3, a_1) \cdot \mid conv(\{0\} \cup A_1 \cup A_2) \mid + det(a_1 + a_2, a_3) \cdot \mid conv(\{0\} \cup A_1) \mid . \quad (16)$$

Set $R(K) := |\Pi K| - (1/4) \cdot |P| \cdot |\Pi P|$. To prove Theorem 2, it suffices to prove that $R(K)$ depends only on the area of P and that equality in Theorem 2 holds for some P.

Let $v_1, ..., v_n$ be the vertices of P. Suppose that P is not a triangle. We may assume that the line segments $[v_1, v_2]$, $[v_2, v_3]$ are the edges A_1, A_2 respectively and that 0 is not contained in the triangle $conv(A_1 \cup A_2)$. The fact that $det(a_i, h_i) > 0$ easily implies that the vector $a_1 + a_2$ and the segment $[v_1, v_3]$ have equal lengths and parallel directions.

We employ here a method often used by Campi, Colesanti and Gronchi (see e.g. [4] or [5]). Consider the family of polygons

$$P_t = conv\{v_1, v_2 + t\nu, v_3, ..., v_n\} \ , \ t \in [t_1, t_2] \ ,$$

where ν is any vector parallel to $a_1 + a_2$ and $[t_1, t_2]$ is the largest interval, in which v_1, v_3 are vertices of P_t for all t in (t_1, t_2). Clearly, (t_1, t_2) contains 0, $P_0 = P$ and, furthermore, the volume of P_t is constant in $[t_1, t_2]$. Also, P_t contains 0 in its interior for all t in $[t_1, t_2]$.

If $A_{1,\ t}, A_{2,\ t}, A_3, ..., A_n$ are the edges of P_t, the corresponding parallel vectors are $a_{1,\ t} = a_1 \pm t\nu$, $a_{2,\ t} = a_2 \mp t\nu$, $a_3, ..., a_n$. Substituting ν by $-\nu$ if necessary, we may assume that $a_{1,\ t} = a_1 - t\nu$ and $a_{2,\ t} = a_2 + t\nu$.

By (12), (14), (15), (16) we have:

$$R\Big(conv(\{e_3\} \cup P_t)\Big) = \sum_{\{i_1,i_2,i_3\} \subseteq [n] \setminus \{1,2\}} |F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |det(x_{i_1}, x_{i_2}, x_{i_3})|$$

$$+ \frac{1}{4} \sum_{\substack{\{i_2,i_3\} \subseteq [n] \setminus \{1,2\} \\ i \in \{1,2\}}} \Big| \ det(a_i + \varepsilon_i t\nu, a_{i_2}) \cdot \mid conv(\{0\} \cup A_{i_3}) \mid$$

$$+ det(a_{i_3}, a_i + \varepsilon_i t\nu) \cdot \mid conv(\{0\} \cup A_{i_2}) \mid + det(a_{i_2}, a_{i_3}) \cdot \mid conv(\{0\} \cup A_{i,t}) \mid \ \Big|$$

$$+ \frac{1}{4} \sum_{i \in [n] \setminus \{1,2\}} \Big| det(a_i, a_1 - t\nu) \cdot \mid conv(\{0\} \cup A_{1,t} \cup A_{2,t}) \mid$$

$$+det(a_1+a_2,a_i)\cdot|\,conv(\{0\}\cup A_{1,t})\,|+det(a_1-t\nu,a_2+t\nu)\cdot|\,conv(\{0\})\cup A_i\,|\,\Bigg|\,,$$

where $\varepsilon_1=-1$, $\varepsilon_2=1$.

Then, $|\,conv(\{0\}\cup A_{1,t}\cup A_{2,t})\,|$ and $det(a_1-t\nu,a_2+t\nu)=2|\,conv\{v_1,v_2+t\nu,v_3\}\,|$ are clearly constant in $[t_1,t_2]$. Also, $det(a_i,a_1-t\nu)$ and $|\,conv(\{0\}\cup A_{i,t})\,|$ are affine in $[t_1,t_2]$, $i=1,2$. As observed previously, each term of the sum above is strictly positive in (t_1,t_2). All these facts imply that the quantity $R(conv(P_t\cup\{e_3\}))$ is affine in $[t_1,t_2]$.

We conclude that for some i,j, with $\{i,j\}=\{1,2\}$,

$$R\Big(conv(P_{t_i}\cup\{e_3\})\Big)\le R(K)=R\Big(conv(P_0\cup\{e_3\})\Big)\le R\Big(conv(P_{t_j}\cup\{e_3\})\Big)\,.$$

It is true that the number of vertices of P_{t_1} and P_{t_2} is strictly less than the number of vertices of P. Thus, by an inductive argument, there exist triangles T_1, T_2 in $\mathbb{R}^2\times\{0\}$ of the same area as P, with:

$$R\Big(conv(T_1\cup\{e_3\})\Big)\le R(K)\le R\Big(conv(T_2\cup\{e_3\})\Big)\,.$$

However, by definition, $R(K)$ is invariant under maps of the form

$$\mathbb{R}^3\ni(s_1,s_2,s_3)\mapsto\ (\Phi(s_1,s_2),s_3)\in\mathbb{R}^3\,,$$

where Φ is an area-preserving, affine transformation on $\mathbb{R}^2$. This shows that $R(K)=R(conv(T\cup\{e_3\}))$, where T is any triangle of the same area as P. Thus, $R(K)$ depends only on the area of P.

In the particular case in which $|P|=1$ and K is the simplex, it is clear that $|P|\cdot|\Pi P|/4=1.5$ and one easily calculates (see e.g. [3]) $|\Pi K|=2$. Thus, $R(K)=1/2$. $\Box$

To prove Corollary 2, take K to be the double cone $conv(P\cup\{\pm e_3\})$ and K' to be the cone $conv(P\cup\{e_3\})$, where P is a centrally symmetric polygon in $\mathbb{R}\times\{0\}$ of area 1. If $F_1,...,F_n$ are the facets of K', that are different from P, by (6), (12) and (14) it follows that:

$$|\Pi K'|=\frac{1}{4}|P|\cdot|\Pi P|+\sum_{\{i_1,i_2,i_3\}\subseteq[n]}|F_{i_1}|\cdot|F_{i_2}|\cdot|F_{i_3}|\cdot|\,det(x_{i_1},x_{i_2},x_{i_3})\,|\,,$$

$$|\Pi K|=2^3\cdot\sum_{\{i_1,i_2,i_3\}\subseteq[n]}|F_{i_1}|\cdot|F_{i_2}|\cdot|F_{i_3}|\cdot|\,det(x_{i_1},x_{i_2},x_{i_3})\,|\,.$$

Thus,

$$\frac{|\Pi K|}{|K|^2}=2^3\cdot\Big(|\Pi K'|-\frac{1}{4}|P|\cdot|\Pi P|\Big)\cdot\frac{9}{4}=9\,,$$

where we used Theorem 2 and the fact that $|K| = 2/3$. $\square$

6. Projection bodies and centroid bodies

Let K be a star body in $\mathbb{R}^d$. The centroid body ΓK of K is defined by its support function

$$h_{\Gamma K}(x) = \int_K | < x, y > | \, dy = \frac{1}{d+1} \int_{S^{d-1}} | < x, y > | \, \rho_K^{d+1}(y) \, dy \ , \ x \in S^{d-1} \ ,$$

where ρ_K is the radial function of K and the last equality follows by integration in polar coordinates. Obviously, ΓK is a zonoid. It can be easily shown that the functional $|\Gamma K|/|K|^{d+1}$ is invariant under non-singular linear transformations. A basic inequality for volumes of centroid bodies is due to Busemann and Petty [1] [13]:

$$\frac{|\Gamma B_1|}{|B_1|^{d+1}} \leq \frac{|\Gamma K|}{|K|^{d+1}} \tag{17}$$

where B_1 is the unit ball. Here, equality holds if and only if K is an origin symmetric ellipsoid. We prove the following:

Proposition 6.1. If K is a star body in $\mathbb{R}^d$, then

$$\frac{|\Pi(\Gamma B_1)|}{|(B_1)|^{(d+1)(d-1)}} \leq \frac{|\Pi(\Gamma K)|}{|K|^{(d+1)(d-1)}} \tag{18}$$

with equality if and only if K is an ellipsoid with center at the origin.

Note here that the quantity $|\Pi(\Gamma K)|/|K|^{(d+1)(d-1)}$ is also invariant under linear maps. Also, as (17) shows, (18) would follow by Petty's conjectured inequality.

In what follows, α_d, β_d etc. will be positive constants that depend only on the dimension d. For $p = 1$ and $p = 2$, define the quantity

$$S_p(K) = \int_{x_1 \in K} \dots \int_{x_d \in K} |det(x_1, ..., x_d)|^p \, dx_1 ... dx_d =$$

$$\frac{1}{(d+p)^d} \int_{x_1 \in S^{d-1}} \dots \int_{x_d \in S^{d-1}} |det(x_1, ..., x_d)|^p \, \rho_K(x_1)^{d+p} ... \rho_K(x_d)^{d+p} \, dx_1 ... dx_d.$$

It is clear that $S_p(K)$ is invariant under volume preserving linear transformations. Also, it follows from (5) that the volume of ΓK is given by

$$|\Gamma K| = \alpha_d S_1(K) \ .$$

We say that K is in isotropic position, if the function

$$S^{d-1} \ni x \mapsto \int_K < x, y >^2 \ dy$$

constant. In this case, the quantity

$$L_K := \frac{(\int_K < x, y >^2 \ dy)^{\frac{1}{2}}}{|K|^{\frac{1}{2}+\frac{1}{d}}}$$

is called the isotropic constant of K. By definition, if K' is an affine image of K then $L_{K'} = L_K$. An obvious fact is that if K is isotropic of volume 1 and $\{e_1, ..., e_d\}$ is an orthonormal basis, then

$$\int_K < x, e_i >< x, e_j > \ dx = L_K^2 \delta_{ij} \ , \ i, j = 1, ..., d \ .$$

It is well known (see e.g. [12]) that there is always a linear transformation T, such that TK is isotropic and of the same volume as K. Thus, it is clear by the above discussion that

$$S_2(K) = \beta_d (L_K^2)^d |K|^{d+2} \ .$$

Lemma 6.1. There exists some constant δ_d, such that if K is a star body in $\mathbb{R}^d$ then

$$S_1(K)^{\frac{d+2}{d+1}} \leq \delta_d S_2(K) \ ,$$

with equality if and only if K is an origin symmetric ellipsoid.

Proof.

$$S_2(K) = \beta_d L_K^{2d} |K|^{d+2} = \beta_d \Big(\frac{1}{d} \int_{TK} |x|^2 \ dx \Big)^d$$

$$= \widetilde{\beta}_d \Big(\int_{S^{d-1}} \rho_{TK}^{d+2}(x) \ dx \Big)^d \geq \beta'_d \Big(\int_{S^{d-1}} \rho_{TK}^{d+1}(x) \ dx \Big)^{d\frac{d+2}{d+1}} \ ,$$

where T is a transformation, such that TK is isotropic of the same volume as K and we used Hölder's inequality in the last part. Equality holds if and only if ρ_{TK} is constant i.e. TK is a ball centered at the origin or, equivalently, K is an ellipsoid with center at 0.

On the other hand, it is clear that there exist some constants γ_d, γ'_d, such that

$$\gamma'_d \int_{S^{d-1}} \rho_{TK}^{d+2}(x) \, dx = \gamma_d \int_{S^{d-1}} \int_{TK} | < x, y > | \, dx \, \frac{dy}{d\omega_d} = V(\Gamma(TK), B_1, ..., B_1) \ ,$$

where ω_d is the volume of the d-dimensional unit ball and $V(\Gamma(TK), B_1, ..., B_1)$ denotes the mixed volume of $\Gamma(TK)$ and B_1 (see [20] for definitions and related inequalities concerning mixed volumes). Now, the Minkowski inequality gives:

$$V(\Gamma(TK), B_1, ..., B_1) \geq |\Gamma(TK)|^{\frac{1}{d}} \cdot |B_1|^{\frac{d-1}{d}} \ ,$$

with equality if and only if $\Gamma(TK)$ is a ball. If TK is a ball centered at the origin, then $\Gamma(TK)$ is a ball. Hence if K is an ellipsoid of center at 0, then equality holds in the last inequality. Combining both inequalities together with the equality cases, we conclude the desired result.□

Proposition 6.1, follows from Lemma 6.1. First we need some additional well known facts. The Busemann formula [1] states:

$$|K|^{\ d-1} = \zeta_d \int_{S^{d-1}} S_1(K \cap x^{\perp}) \ dx \ . \tag{19}$$

Using a generalization of Busemann's formula Weil [22] showed that if

$$h_Z(x) = \frac{1}{2} \int_{S^{d-1}} | < x, y > | \ f(y) \ dy \ , \ x \in \mathbb{R}^d \ ,$$

is the support function of a d-dimensional zonoid Z, for some measurable function $f : S^{d-1} \to \mathbb{R}$, then its surface area measure is absolutely continuous with respect to the Lebesque measure and its density function f_Z is given by:

$$f_Z(x) = \theta_d \int_{S^{d-1} \cap x^{\perp}} .. \int_{S^{d-1} \cap x^{\perp}} det(x_1, ..., x_{d-1})^2 \ f(x_1)...f(x_{d-1}) \ dx_1...dx_{d-1} \ .$$

Let P be a convex body, having absolutely continuous surface area measure with density f. Petty [14] proved the following inequality

$$|\Pi P| \geq \overline{c_d} \Big(\int_{S^{d-1}} f^{\frac{d}{d+1}}(x) \ dx \Big)^{d+1} \ , \tag{20}$$

with equality if and only if P is an ellipsoid. The quantity $\Omega(P) := \int_{S_{d-1}} f^{d/d+1} dx$ is called the affine surface area of P.

Proof of Proposition 6.1.
Note that (according to Weil's result) the surface area measure of ΓK is absolutely continuous with respect to the Lebesque measure and its density is given by:

$$f(x) = l_d \cdot S_2(K \cap x^{\perp}) \ , \ x \in S^{d-1} \ .$$

Thus, by (20), Lemma 6.1 and (19), we have

$$|\Pi(\Gamma K)| \geq \lambda_d \Big(\int_{S^{d-1}} S_2(K \cap x^{\perp})^{\frac{d}{d+1}} \; dx \; \Big)^{d+1}$$

$$\geq \lambda'_d \Big(\int_{S^{d-1}} S_1(K \cap x^{\perp}) \; dx \Big)^{d+1} = \lambda''_d |K|^{(d-1)(d+1)} \; .$$

Equality holds in both inequalities if K is an ellipsoid with center at 0 and in the second one only if $K \cap x^{\perp}$ is an ellipsoid centered at the origin, for every $x \in S^{d-1}$. It follows then from [6], Theorem 7.1.5 that K is necessarily an ellipsoid centered at the origin. □

Next we state an application of Proposition 6.1 in three dimensions.

Theorem 4. Let $n \geq 3$ be an integer. Among all 3-dimensional convex bodies of volume 1, ellipsoids with center at the origin are exactly the bodies that minimize the functional

$$Q_n(K) := \int_{x_1 \in K} \dots \int_{x_n \in K} \Big| \Pi \Big(\sum_{i=1}^{n} [-x_i, x_i] \Big) \Big| \; dx_1 \dots dx_n \; .$$

In other words, the mean value of the volume of the projection body of the sum of n line segments picked uniformly and independently from a convex body K of prescribed volume, is minimal if and only if K is an origin symmetric ellipsoid. Theorem 4 is formally related to Petty's conjecture as follows: If one could replace the Minkowski sum with the convex hull in Theorem 4, then Petty's conjecture would be correct in three dimensions. Note, also, that the functional $Q_n(K)/|K|^3$ is invariant under non-singular linear transformations.

Proof of Theorem 4.

Let us use (8) and Lemma 3.3 to derive a simple expression for $Q_n(K)$:

$$Q_n(K) = \alpha_1 \binom{n}{3} \int_{(x_1,x_2,x_3) \in K^3} det(x_1, x_2, x_3)^2 \; dx_1 dx_2 dx_3$$

$$+\alpha_2 \binom{n}{4} \int_{(x_1,x_2,x_3,x_4) \in K^4} |det(x_1, x_2, x_3) \cdot det(x_1, x_2, x_4)| \; dx_1 dx_2 dx_3 dx_4$$

$$+\alpha_3 \binom{n}{5} \int_{(x_1,x_2,x_3,x_4,x_5) \in K^5} |det(x_1, x_2, x_3) \cdot det(x_1, x_4, x_5)| \; dx_1 dx_2 dx_3 dx_4 dx_5$$

$$+\alpha_4 \binom{n}{6} \int_{(x_1,\dots,x_6)\in K^6} |det(x_1 \wedge x_2, x_3 \wedge x_4, x_5 \wedge x_6)| \; dx_1 \dots dx_6$$

$$=: \alpha_1 \binom{n}{3} A_1(K) + \alpha_2 \binom{n}{4} A_2(K) + \alpha_3 \binom{n}{5} A_3(K) + \alpha_4 \binom{n}{6} A_4(K) \; ,$$

where $\alpha_1, \dots, \alpha_4$ are absolute positive constants.

Since $A_1(K) = S_2(K) = \beta_3 (L_K^2)^3$ and since it is well known that the isotropic constant is minimal if and only if K is an ellipsoid centered at 0, it follows that

$$A_1(K) \geq A_1(B_1) \; ,$$

where B is the ball of center 0 and volume 1, with equality if and only if K is an ellipsoid centered at 0.

Also, by (7) and Proposition 6.1, we have:

$$A_4(K) = \mu |\Pi(\Gamma K)| \geq \mu |\Pi(\Gamma B_1)| = A_4(B_1) \; ,$$

where μ is an absolute constant.

To complete the proof of Theorem 4, observe that $A_2(K) =$

$$\int_{x_1\in K} \int_{x_2\in K} \Big(\int_{x_3\in K} |det(x_1, x_2, x_3)| \; dx_3 \cdot \int_{x_4\in K} |det(x_1, x_2, x_4)| \; dx_4 \Big) \; dx_1 dx_2$$

$$= \int_{x_1\in K} \int_{x_2\in K} \Big(\int_{x_3\in K} |det(x_1, x_2, x_3)| \; dx_3 \Big)^2 \; dx_1 dx_2$$

and, similarly,

$$A_3(K) = \int_{x_1\in K} \Big(\int_{x_2\in K} \int_{x_3\in K} |det(x_1, x_2, x_3)| \; dx_2 dx_3 \Big)^2 \; dx_1 \; .$$

One may, now, use the process of Steiner symmetrization in a standard way (taking, also in advantage the obvious fact that $A_2(K)$, $A_3(K)$ are invariant under volume-preserving linear maps) to show that

$$A_i(K) \geq A_i(B_1) \; , \; i = 2, 3 \; .$$

We omit the details. $\square$

7. Some concluding remarks

§1) The fact that the class of three-dimensional zonoids, in which equality holds in (2), strictly contains the class of cylinders is of some interest. For instance, one can easily derive an infinite family of counterexamples to Schneider's conjecture in three dimensions, also extended to any dimension using (4).

Indeed, suppose that Z is a zonotope in $\mathbb{R}^3$, which is the sum of five line segments in general position or the sum of a cylinder and a line segment, but not a cylinder. Then,

$$|\Pi(\Pi^{-1}Z)| > 2^3|\Pi^{-1}Z|^2 .$$

To see this, note that as shown in Schneider [20] (p. 417),

$$\frac{|\Pi(\Pi Z)|}{|\Pi Z|^2} \leq \frac{|\Pi Z|}{|Z|^2} , \tag{21}$$

with equality if and only if $\Pi(\Pi Z)$ and Z are homothetic. However, Weil [21] showed that cylinders are the only three-dimensional polytopes having this property. Since Z is a polytope, but not a cylinder, so is $\Pi^{-1}Z$. Thus, replacing in (21) Z by $\Pi^{-1}Z$ and using the equality characterization in Theorem 1, the conclusion follows. As an example, one may conclude that for every centrally symmetric three-dimensional polytope K of volume 1, with at most ten facets, it is true that $|\Pi K| \geq 8$, with equality if and only if K is a cylinder.

If E is a three-dimensional ellipsoid, one may easily compute that $|\Pi E|/|E|^2 < 8$. Thus, if K is a cone, a centrally symmetric double cone or of the form $\Pi^{-1}Z$, where Z is a zonoid for which equality in (2) is attained, then $|\Pi E|/|E|^2 < |\Pi K|/|K|^2$. As far as we know, there are no other natural three-dimensional classes of convex bodies, beside cylinders, in which Petty's conjecture has been confirmed.

§2) The problem of proving the analogous to (2) in all dimensions seems to be much more complicated than the three-dimensional case. One of the main reasons, in our opinion, is that it is not very clear what someone would expect as equality cases. An inequality analogous to (10) would be the key to this problem. One may consider the functions S and T in d dimensions, defined by (9) and (8) respectively, to conclude that the inequality $|\Pi Z|/|Z|^{d-1} \leq 2^d$ holds for any zonoid Z, if and only if the same inequality holds for zonotopes being the sum of at most $d(d-1)$-line segments.

§3) If K is a 3-dimensional star body, set

$$P_n(K) := \int_{x_1\in K} .... \int_{x_n\in K} |\sum_{i=1}^{n}[-x_i, x_i]|^2 \ dx_1...dx_n \ .$$

To conclude this paper, we would like to remark that the value of the ratio $Q_n(B)/P_n(B)$ agrees with Petty's conjecture for all integers $n \geq 3$, where B is a three-dimensional euclidian ball with center at 0. Indeed, one computes

$$P_n(K) = \beta_1 \binom{n}{3} A_1(K) + \beta_2 \binom{n}{4} A_2(K) + \beta_3 \binom{n}{5} A_3(K) + \beta_4 \binom{n}{6} B_4(K),$$

where

$$B_4(K) := \Big(\int_{x_1\in K}\int_{x_2 \ \in K}\int_{x_3\in K} |det(x_1, x_2, x_3)| \ dx_1dx_2dx_3\Big)^2$$

and $\beta_1, ..., \beta_4$ are absolute constants. Taking n to be 3, 4, 5 successively, it follows by Theorem 1 that $\alpha_i/\beta_i = 8 > |\Pi B|/|B|^2$, $i = 1, 2, 3$. Also, the law of large numbers implies easily that the random zonotope $\frac{1}{n}\sum_{i=1}^{n}[-x_i, x_i]$ converges almost surely to a multiple of the centroid body of K, as n tends to infinity, thus $lim_{n\to\infty}Q_n(K)/P_n(K) = |\Pi(\Gamma K)|/|\Gamma K|^2$. On the other hand, it is clear that $lim_{n\to\infty}Q_n(K)/P_n(K) = \alpha_4 A_4(K)/\beta_4 B_4(K)$. Taking $K = B$, it follows that $\alpha_4 A_4(B)/\beta_4 B_4(B) = |\Pi B|/|B|^2$, so $Q_n(B)/P_n(B) > |\Pi B|/|B|^2$, $n \geq 3$. The last inequality would follow by Petty's conjectured inequality.

Acknowledgement. Special thanks to Prof. Souzana Papadopoulou for significant contribution in this paper.

References

[1] Busemann, H., Volume in terms of concurrent sections, *Pacific J. Math.*, **3** (1953), 1-12.

[2] Brannen, N., S., Volumes of projection bodies, *Mathematika*, **43** (1996), 255-264.

[3] Brannen, N., S., Three-dimensional projection bodies, *Adv. Geom.*, **5** (2005), 1-13.

[4] Campi, S., Colesanti A. and Gronchi, P., A note on Sylvester's problem for random polytopes in a convex body, *Rendiconti Ist Matematica dell'Universita Trieste*, **31** (1999), 79-94.

[5] Campi, S. and Gronchi, P., Extremal convex sets for Sylvester-Busemann type functionals, *Applicable Analysis*, **85** (2006), 129-141.

[6] Gardner, R. J., Geometric Tomography, *Encyclopedia of Mathematics and its Applications* 58, Cambridge University Press, Cambridge, 1995.

[7] Lutwak, E., On a conjectured projection inequality of Petty. *Contemp. Math.*, **113** (1990), 171-182.

[8] Lutwak, E., On quermassintegrals of mixed projection bodies, *Geom. Dedicata*, **33** (1990), 51-58.

[9] Lutwak, E., Yang, D. and Zhang G., A new affine invariant for polytopes and Schneider's projection problem, *Trans. Amer. Math. Soc.* **353** (2001), 1767-1779.

[10] Martini, H. and Mustafaev, Z., On Isoperimetric Inequalities in Minkowski Spaces, *Journal of Inequalities and Applications*, vol. 2010, Article ID 697954, 18 pages (2010).

[11] Makai, E. Jr. and Martini, H., The cross-section body, plane sections of convex bodies and approximation of convex bodies. I, *Geometriae Dedicata*, **63** (1996), 267-296.

[12] Milman, V.D. and Pajor, A., Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, *Geometric Aspects of Functional Analysis*, Lecture Notes in Math., **1376** (1989), Springer Berlin, 64-104.

[13] Petty, C. M., Centroid surfaces, *Pacific J. Math.*, **11** (1961), 1535-47.

[14] Petty, C. M., Projection bodies, *Proc. Coll. Convexity, Copenhagen, 1965, Kobenhavns Univ. Mat. Inst.*, 1967, 234-241.

[15] Petty, C. M., Isoperimetric problems, *Proc. Conf. Convexity Combinat. Geom., Univ. Oklahoma*, 1971, 26-41.

[16] Reisner, S., Random polytopes and the volume product of symmetric convex bodies, *Math. Scand.*, **57** (1985), 386-392.

[17] Reisner, S., Zonoids with minimal volume product, *Math. Z.*, **192** (1986), 339-346.

[18] Rogers, C. A. and Shephard, G. C., The difference body of a convex body, *Arch. Math.*, **8** (1957), 220-233.

[19] Schneider, R., Random hyperplanes meeting a convex body. *Z. Wahrscheinlichkeitsth. verw. Geb.*, **61** (1982), 379-387.

[20] Schneider, R., Convex Bodies: The Brunn-Minkowski theory, *Encyclopedia of Mathematics and its Applications* 44, Cambridge University Press, Cambridge, 1993.

[21] Weil W., Über die Projektionenkörper konvexer Polytope, *Arch. Math.*, **22** (1971), 664-672.

[22] Weil W., Kontinuierliche Linearkombination won Strecken, *Math. Z.*, **148** (1976), 71-84.

[23] Zhang, G., Restricted chord projection and affine inequalities, *Geom. Dedicata*, **39** (1991), 213-222.