Visualization of Graphs

Lecture 9:
Beyond Planarity
Drawing Graphs with Crossings

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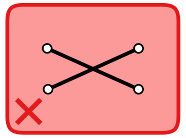
The slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, U. Trier

Planar Graphs

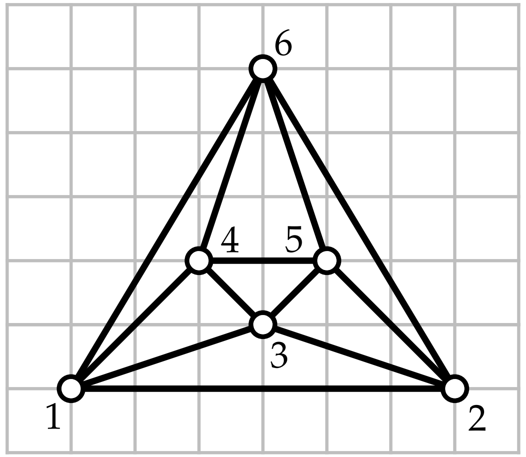
Planar graphs admit drawings in the plane without crossings.



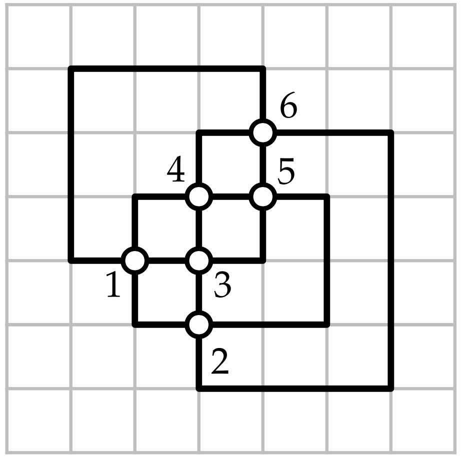
Plane graph is a planar graph with a plane embedding = rotation system.

Recognizable in linear time

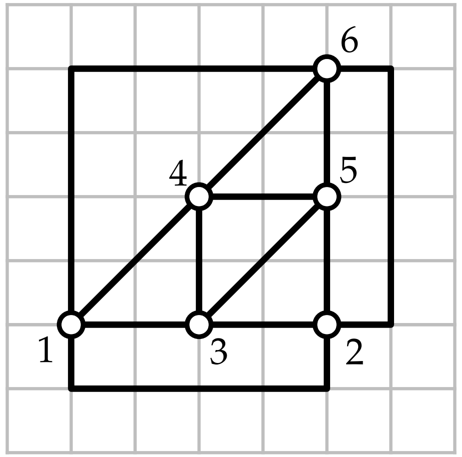
Different drawing styles...



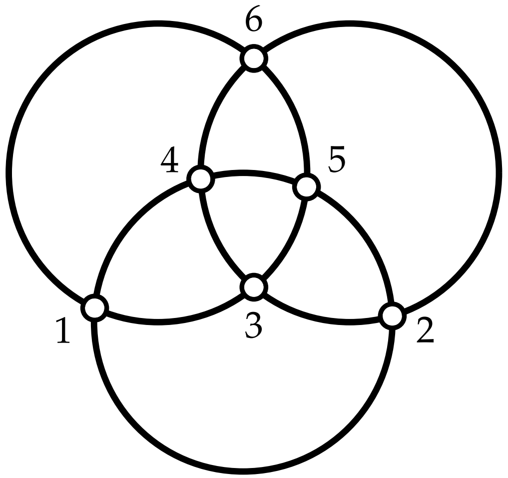
straight-line drawing



orthogonal drawing



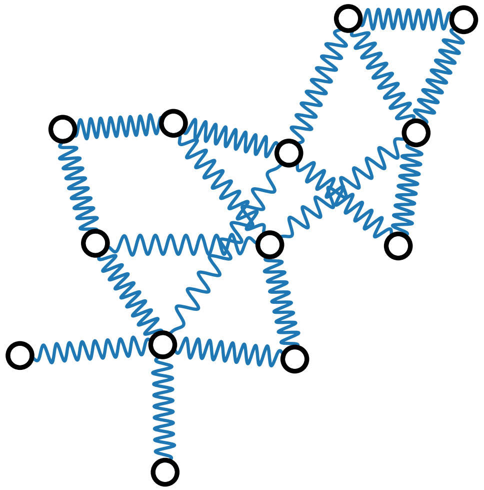
grid drawing with bends & 3 slopes



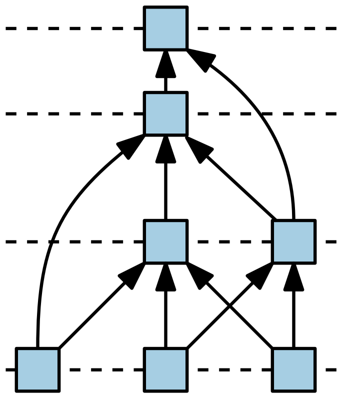
circular-arc drawing

And Non-Planar Graphs?

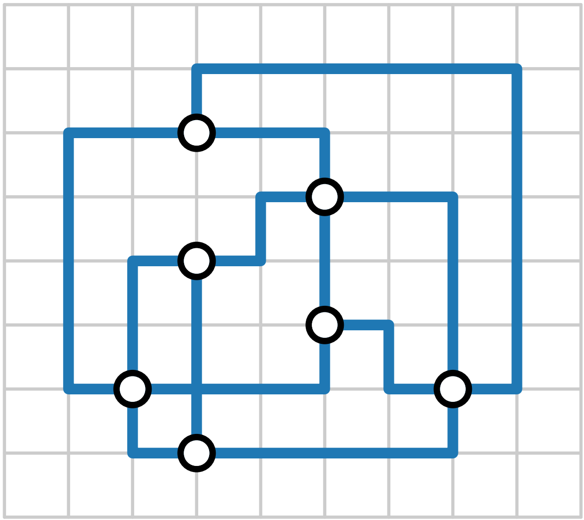
We have seen a few drawing styles:



force-directed drawing

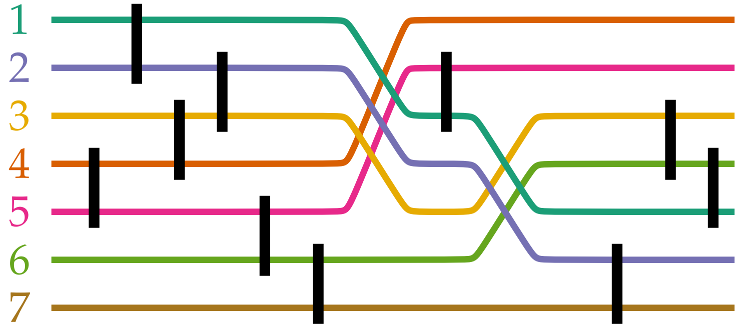


hierarchical drawing

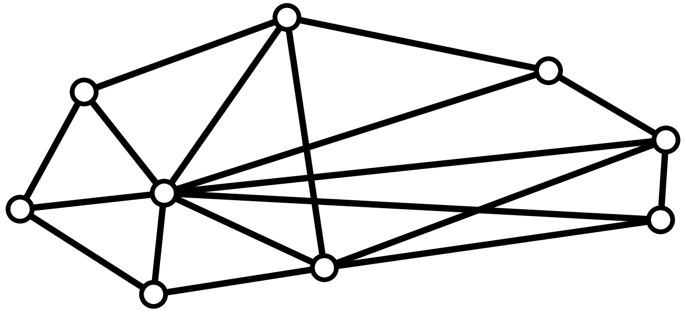


orthogonal layouts
(via planarization)

Maybe not all crossings are equally bad?



block crossings



Which crossings feel worse?

Eye-Tracking Experiment

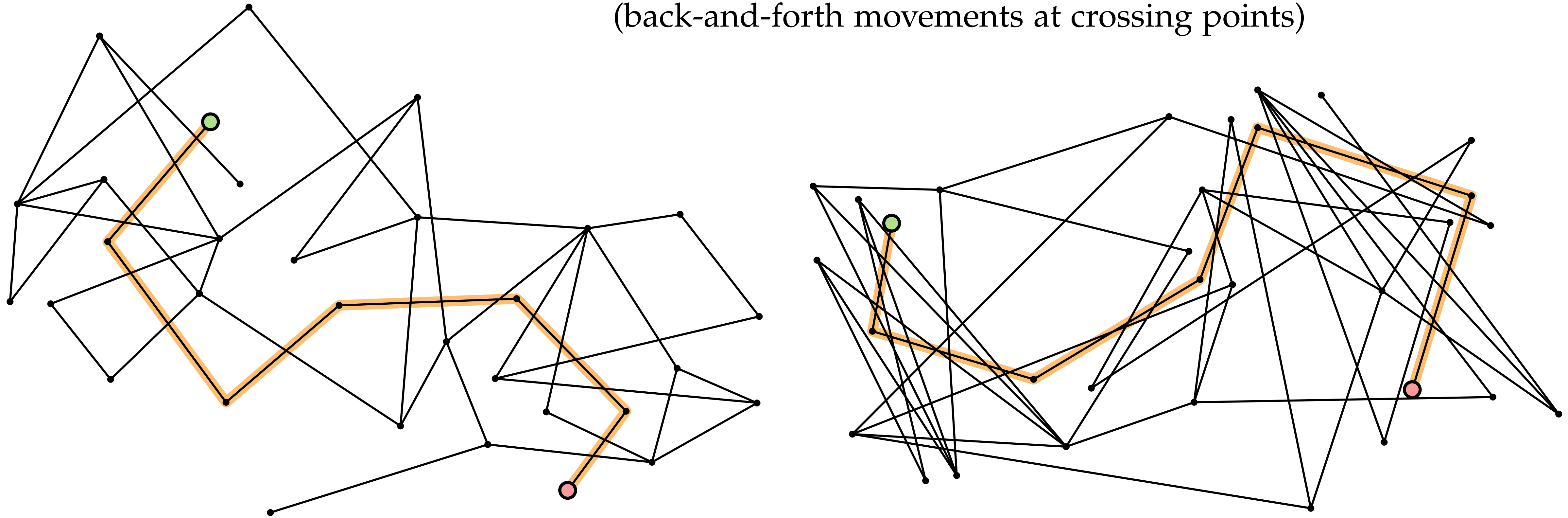
[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

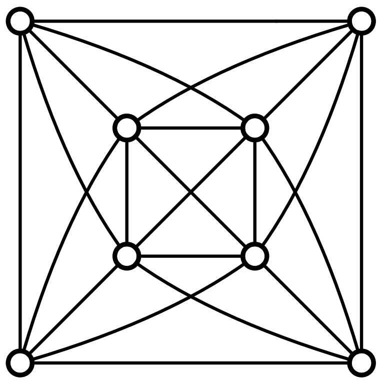
Results:

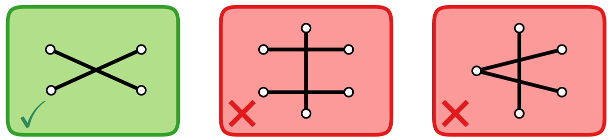
no crossings	eye movements smooth and fast
large crossing angles	eye movements smooth but slightly slower
small crossing angles	eye movements no longer smooth and very slow (back-and-forth movements at crossing points)



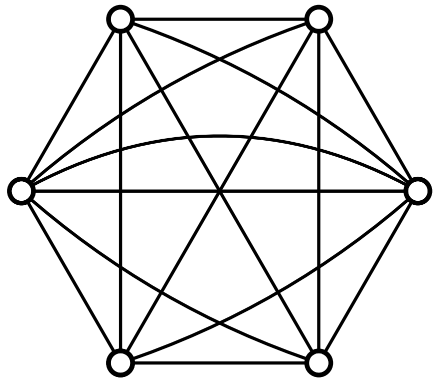
Some Beyond-Planar Graph Classes

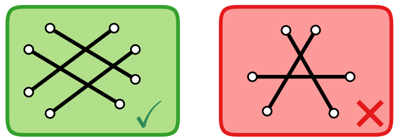
We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.



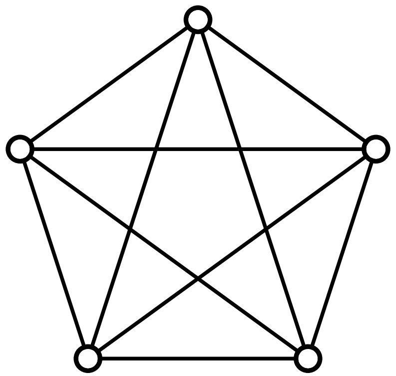


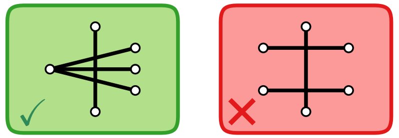
k-planar ($k = 1$)



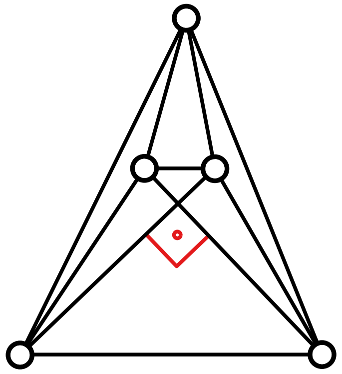


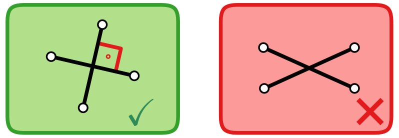
k-quasi-planar ($k = 3$)





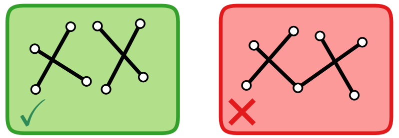
fan-planar



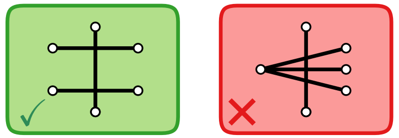


RAC
right-angle crossing

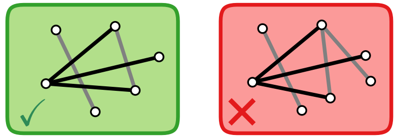
There are many more beyond planar graph classes. . .



IC (independent crossing)



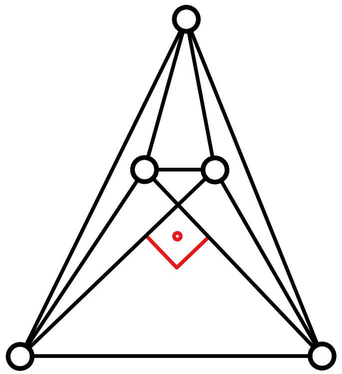
fan-crossing-free



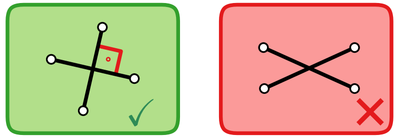
skewness-k ($k = 2$)

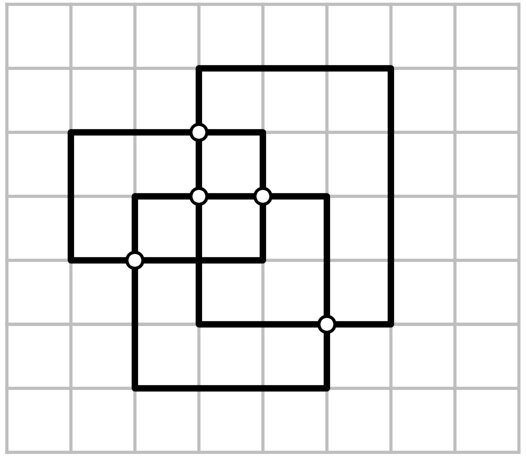
combinations, . . .

Drawing Styles for Crossings

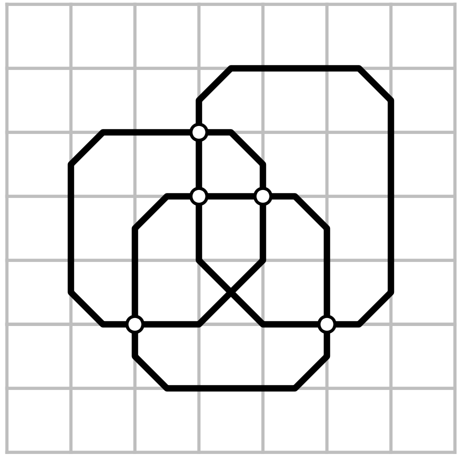


RAC
right-angle crossing

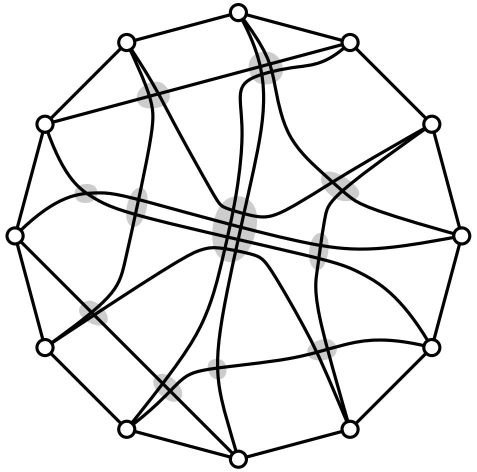




orthogonal

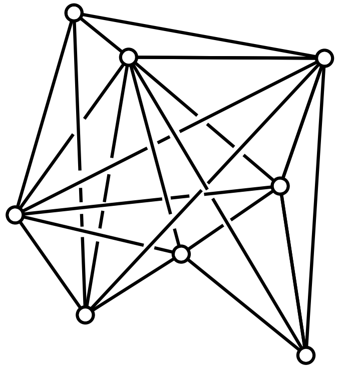


slanted orthogonal

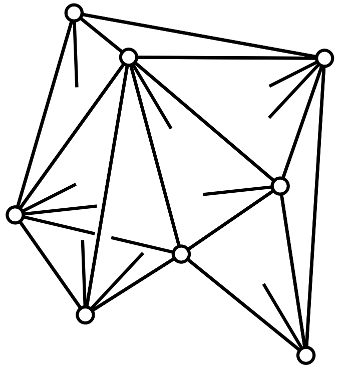


block/bundle crossings

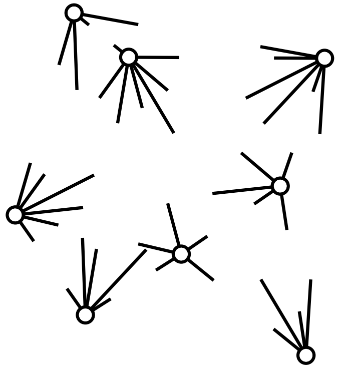
circular layout: 28 invididual
vs. 12 bundle crossings



cased crossings

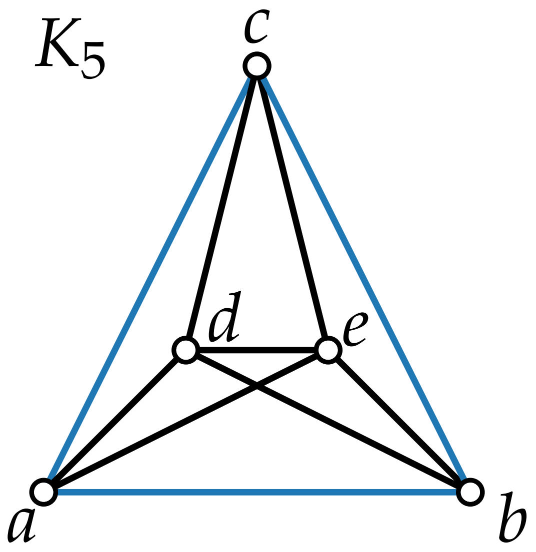


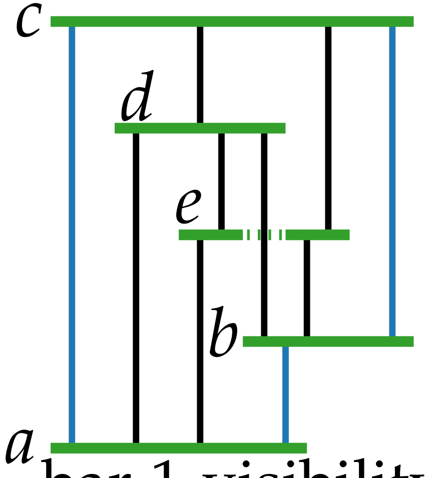
sym. partial
edge drawing



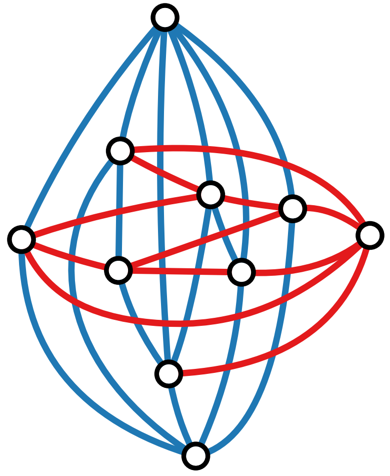
1/4-SHPED

Geometric Representations

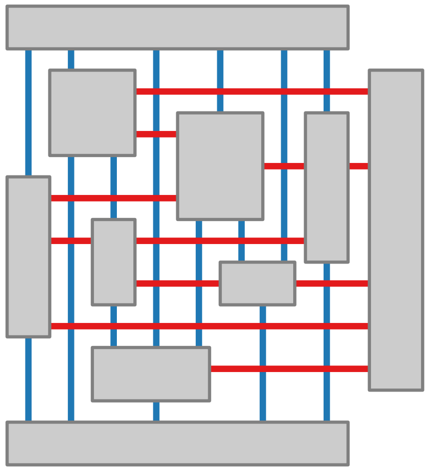




bar 1-visibility representation (B1VR)



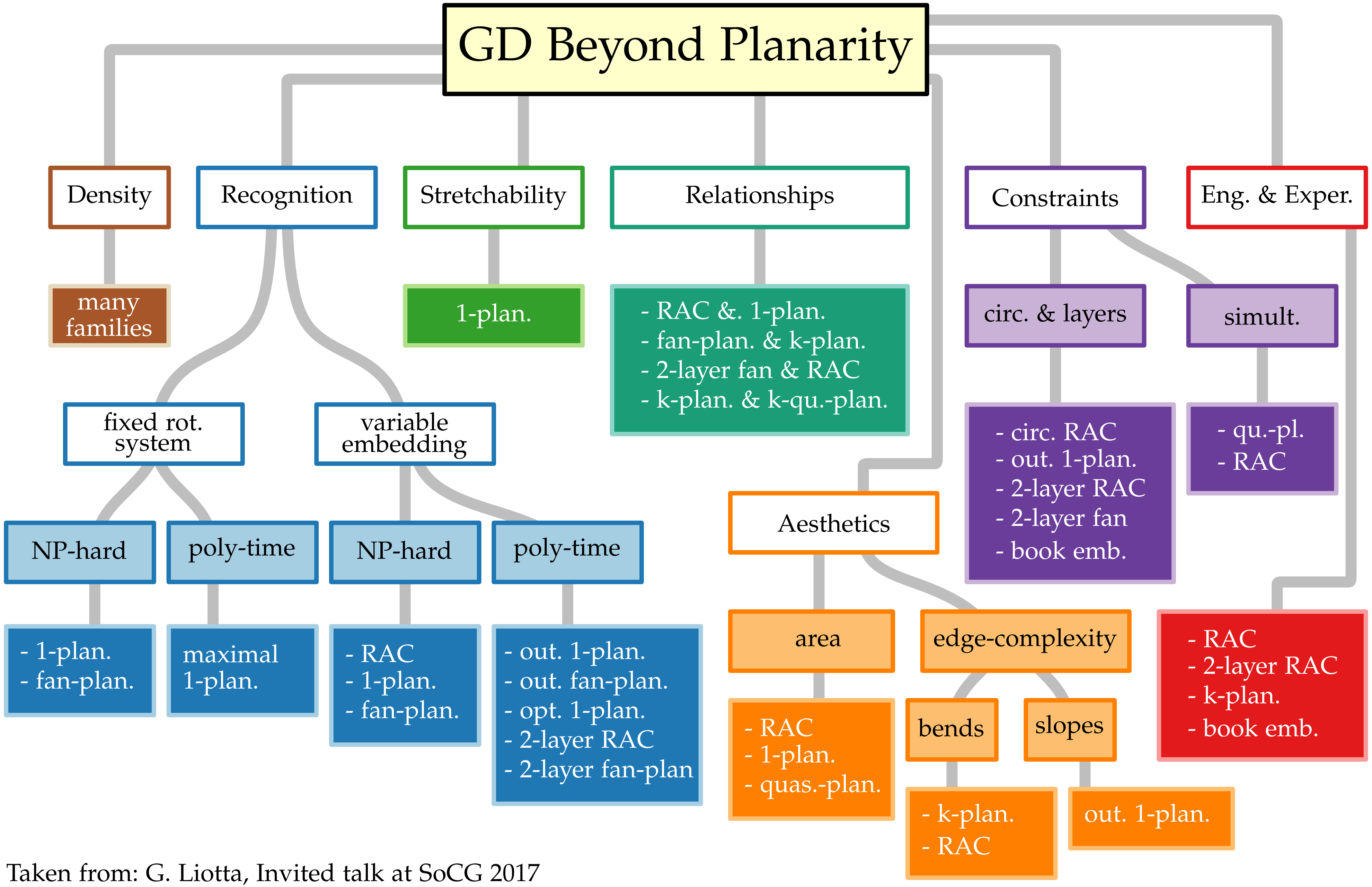
thickness two graph



rectangle visibility representation

- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
- G has at most $6n - 20$ edges [Bose et al. 1997]
- Recognition is NP-complete [Shermer 1996]
- Recognition becomes polynomial if embedding is fixed [Biedl et al. 2018]

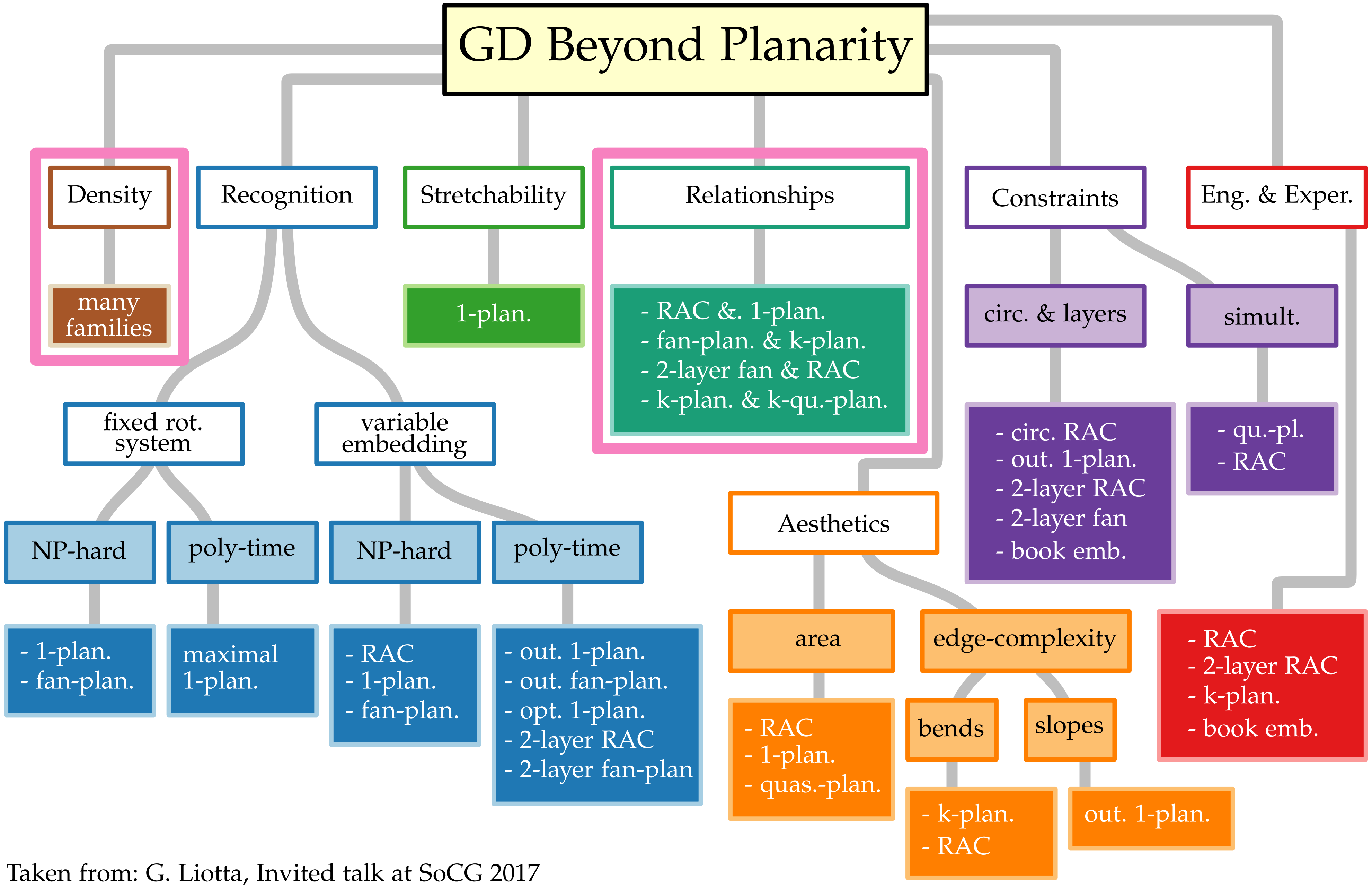
GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Part II:
Density & Relationships

GD Beyond Planarity: a Taxonomy



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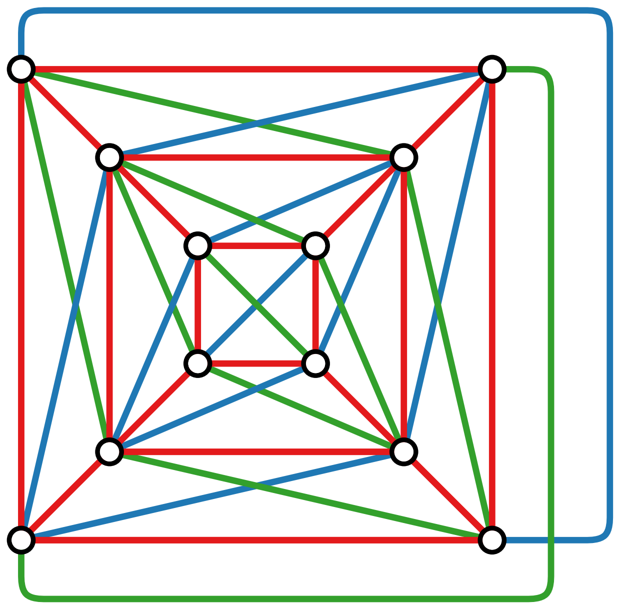
Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n-8$ edges, which is a tight bound.



Planar structure:
$2n-4$ edges
$n-2$ faces
Edges per face: 2 edges
Total: $4n-8$ edges

Proof sketch.

- red edges do not cross
- each blue edge crosses a green edge
- red-blue plane graph G_{rb}

  $m_{rb} \leq 3n-6$
- green plane graph G_g

  $m_g \leq 3n-6 \quad \Rightarrow \quad m \leq m_{rb} + m_g \leq 6n-12$

Observe that each green edge joins two faces in G_{rb}.

$$m_g \leq f_{rb}/2 \leq (2n-4)/2 = n-2$$

$$\Rightarrow \quad m = m_{rb} + m_g \leq 3n-6+n-2 = 4n-8$$

Density of 1-Planar Graphs

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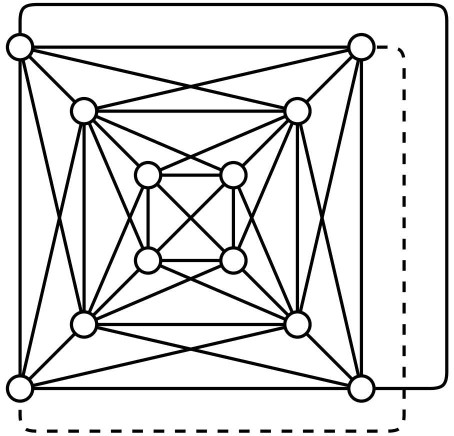
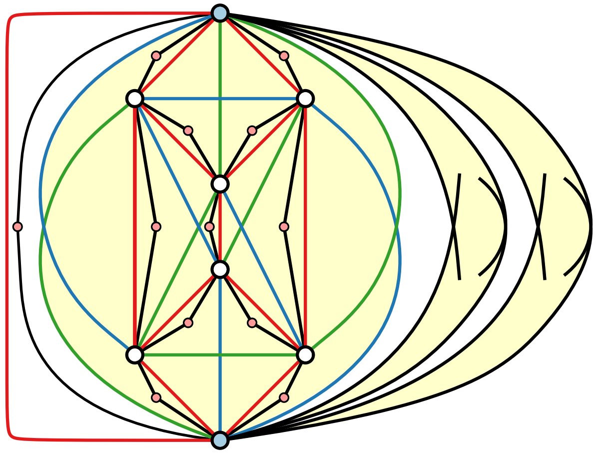


A 1-planar graph with n vertices is called **optimal** if it has exactly $4n - 8$ edges.
A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.



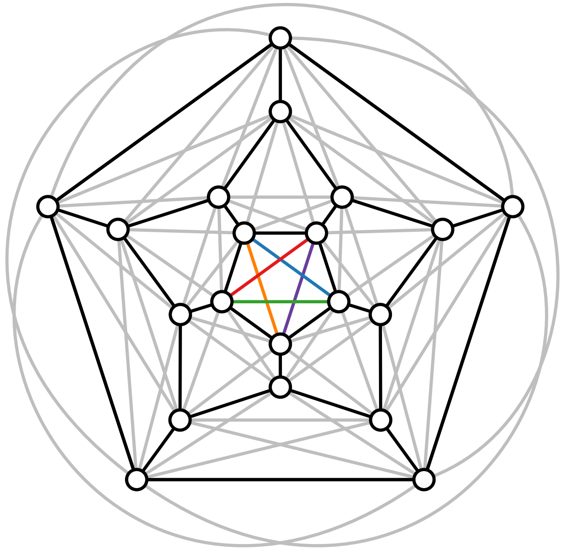
Theorem. [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with n vertices and $\underbrace{45/17n}_{\approx 2.65n} - O(1)$ edges.

Theorem. [Didimo 2013]
A 1-planar graph with n vertices that admits a **straight-line drawing** has at most $4n - 9$ edges.

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]



optimal 2-planar

$n - m + f = 2$
$m = c \cdot f$?
$m = \frac{5}{2} f$

Planar structure:
$\frac{5}{3}(n-2)$ edges
$\frac{2}{3}(n-2)$ faces
Edges per face: 5 edges
Total: $5(n-2)$ edges

Density of k-Planar Graphs

Theorem.
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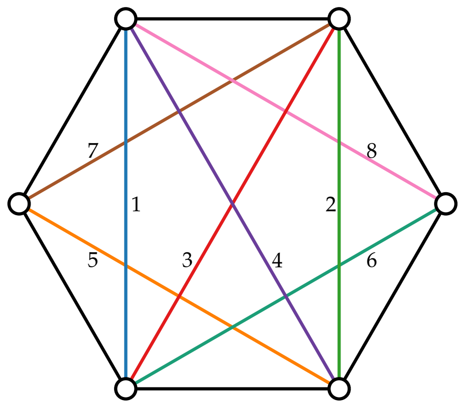
k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]



optimal 3-planar

Density of k-Planar Graphs

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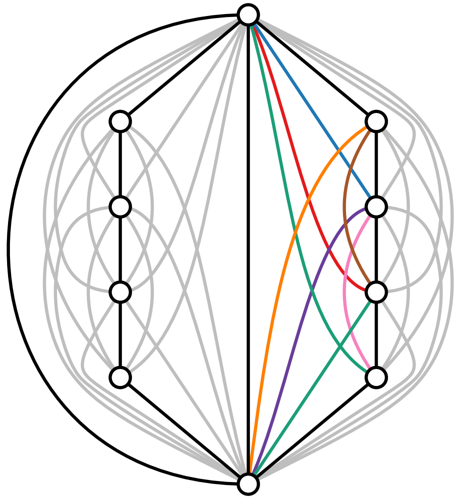


optimal 3-planar

Planar structure:
$\frac{3}{2}(n-2)$ edges
$\frac{1}{2}(n-2)$ faces

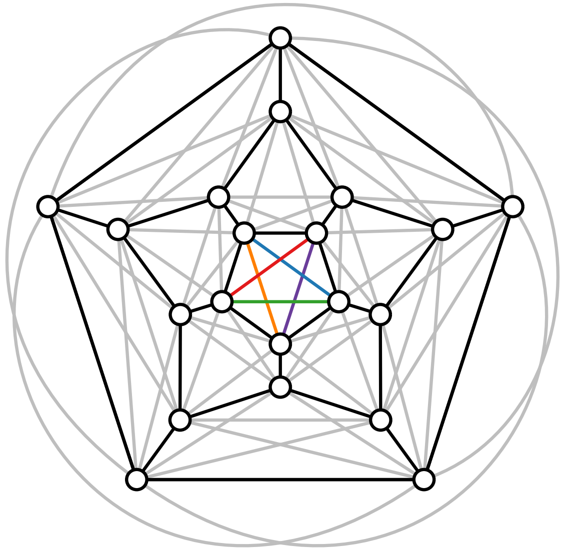
Edges per face: 8 edges

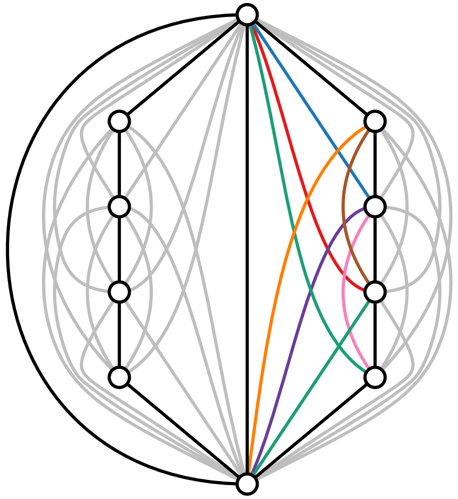
Total: $5.5(n-2)$ edges

Density of k-Planar Graphs

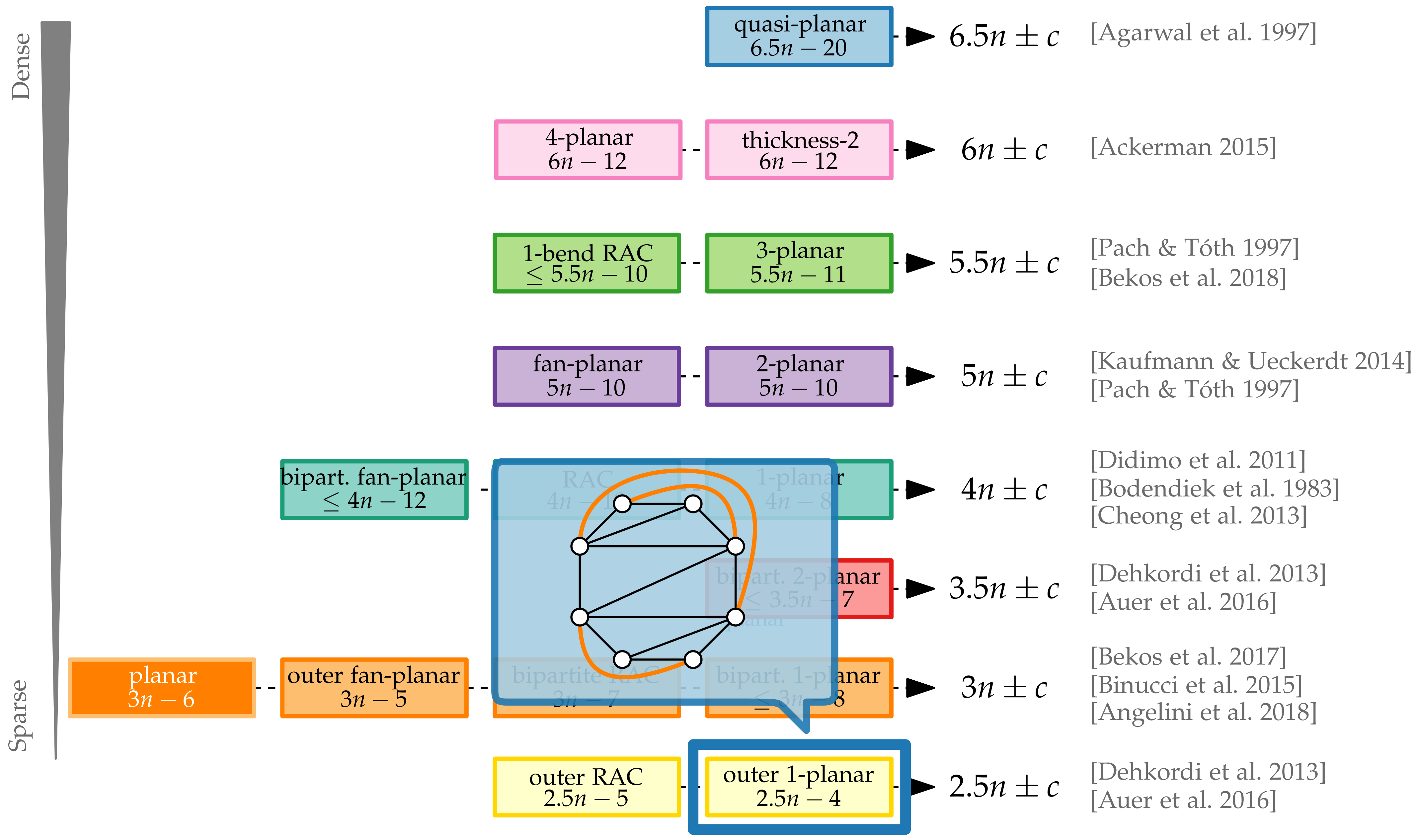
Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]
4	$6(n-2)$	[Ackerman 2015]
> 4	$4.108\sqrt{kn}$	[Pach and Tóth 1997]


optimal 2-planar


optimal 3-planar

GD Beyond Planarity: a Hierarchy



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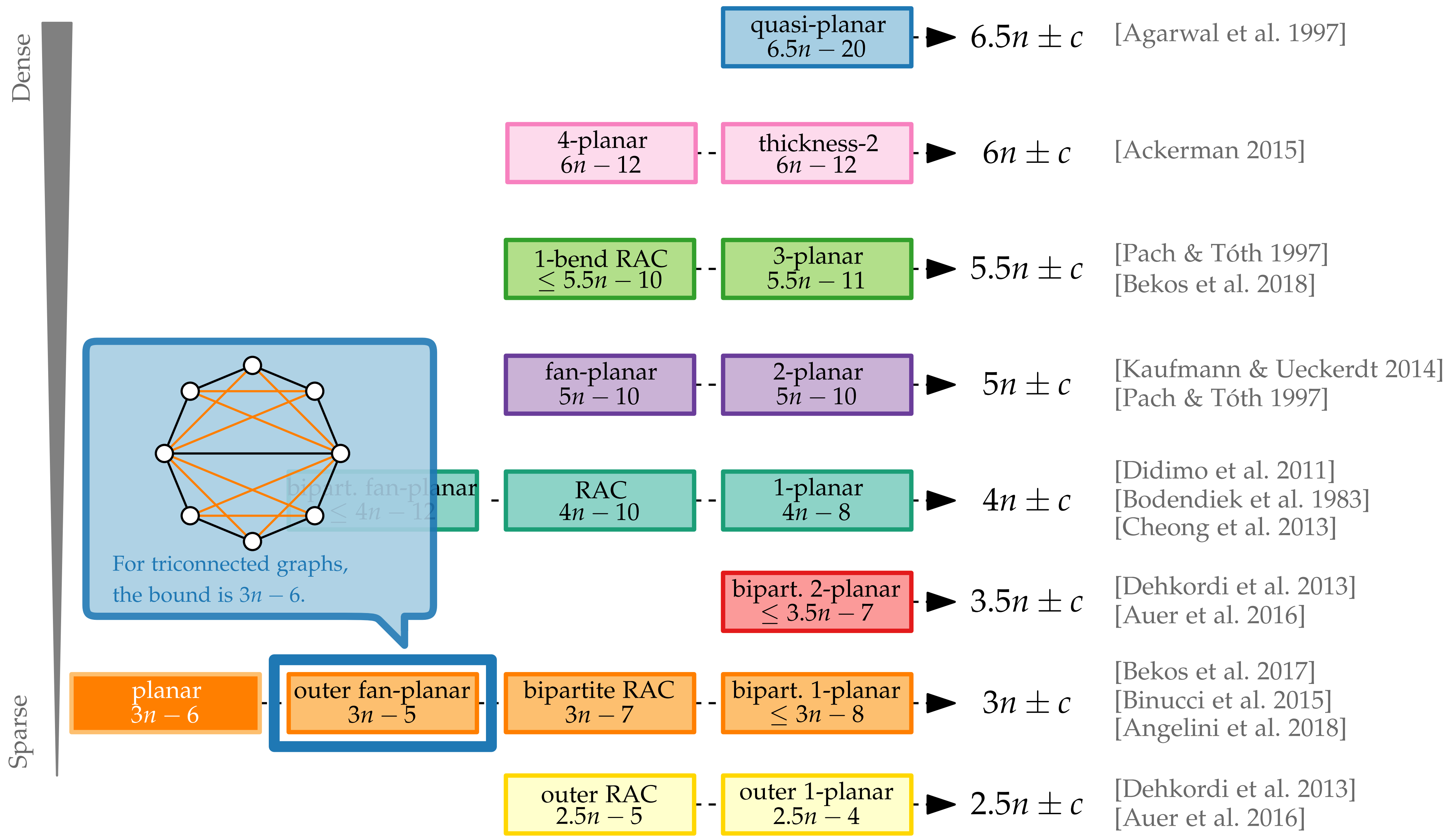
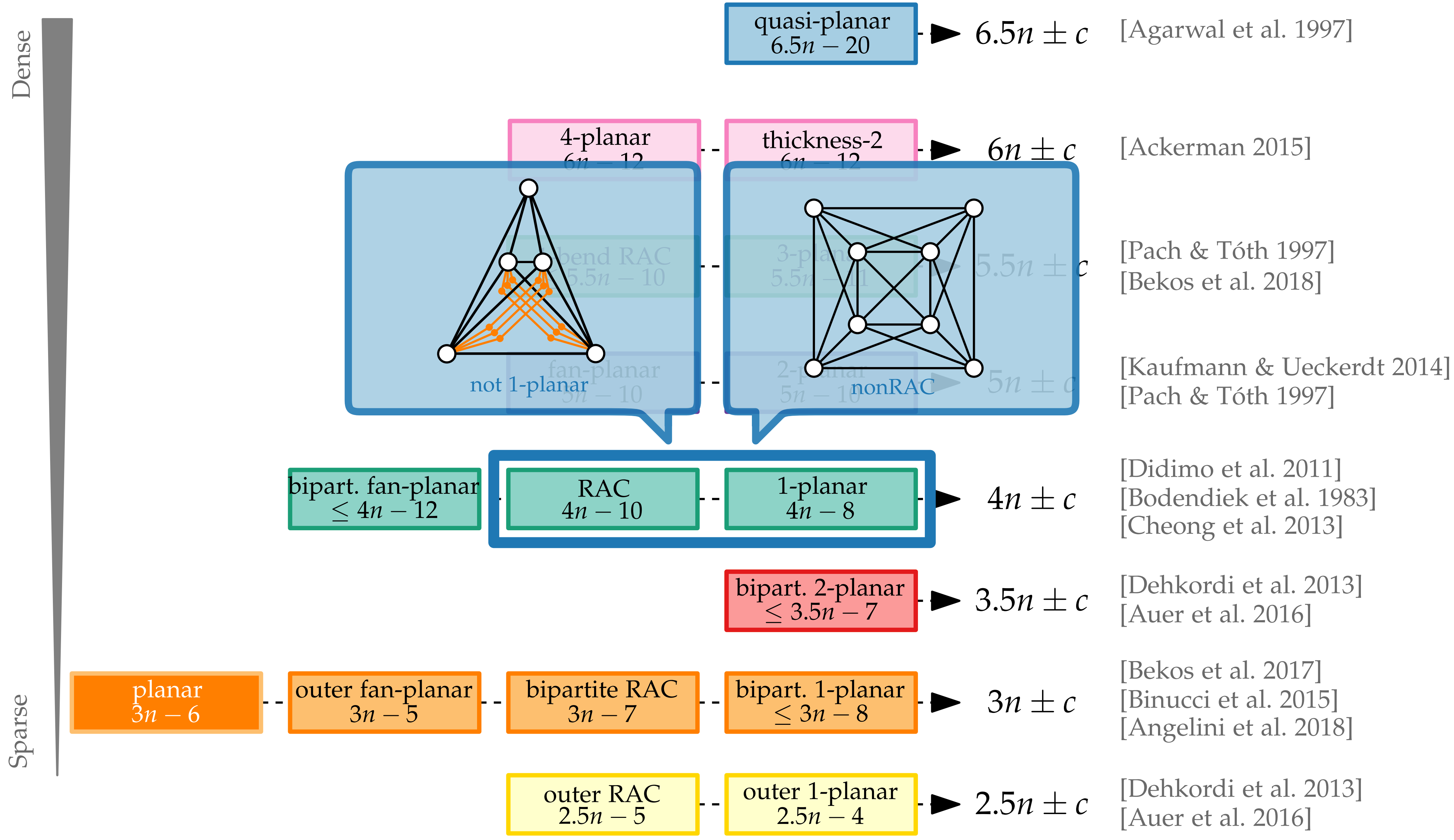


GD Beyond Planarity: a Hierarchy



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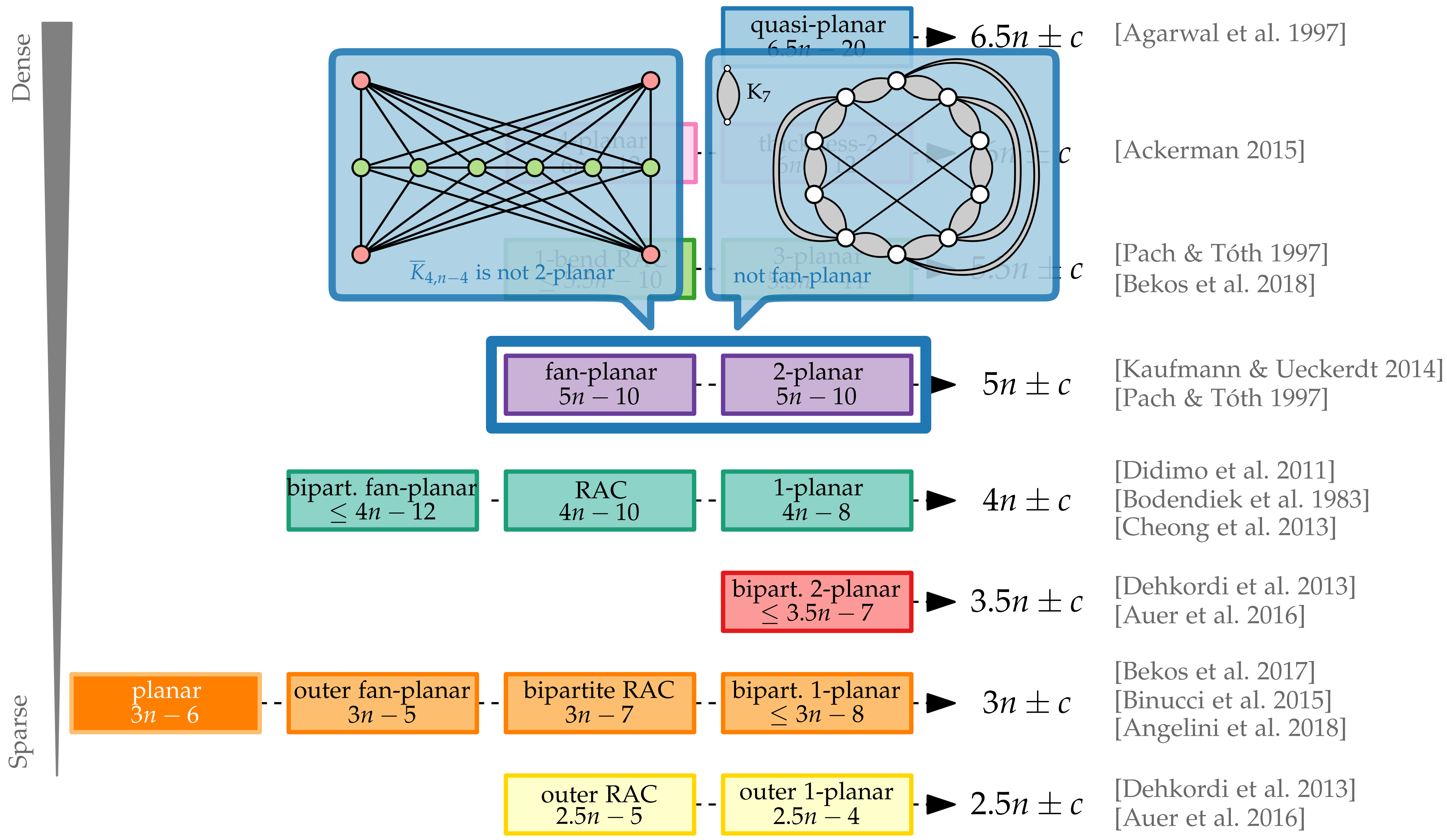
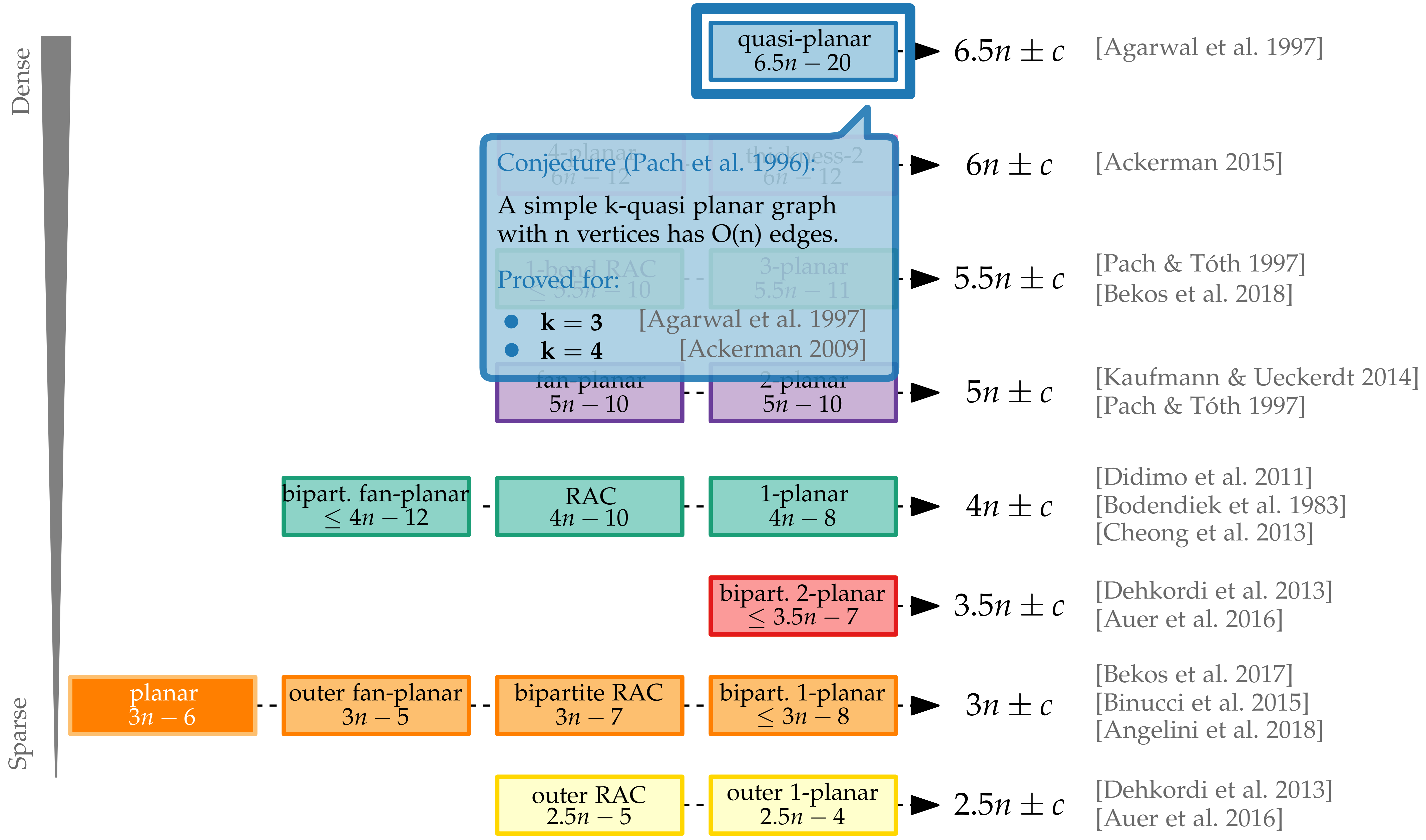


GD Beyond Planarity: a Hierarchy

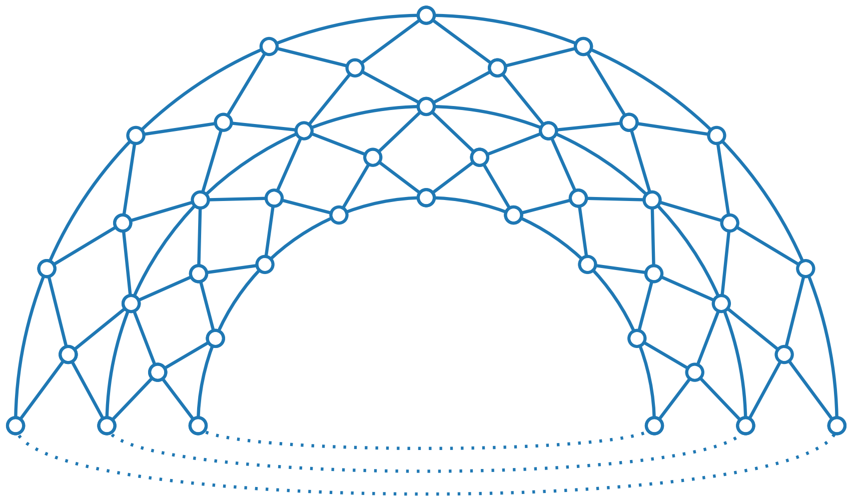


Crossing Numbers

The ***k*-planar crossing number** $\mathrm{cr}_{k\text{-pl}}(G)$ of a graph G is the number of crossings required in any k-planar drawing of G.

- $\mathrm{cr}_{1\text{-pl}}(G) \geq n - 2$
- $\mathrm{cr}(G) = 1 \Rightarrow \mathrm{cr}_{1\text{-pl}}(G) = 1$

Theorem. [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph G with $n = 11\ell + 2$ vertices such that $\mathrm{cr}(G) = 2$ and $\mathrm{cr}_{1\text{-pl}}(G) = n - 2$.



Crossing Numbers

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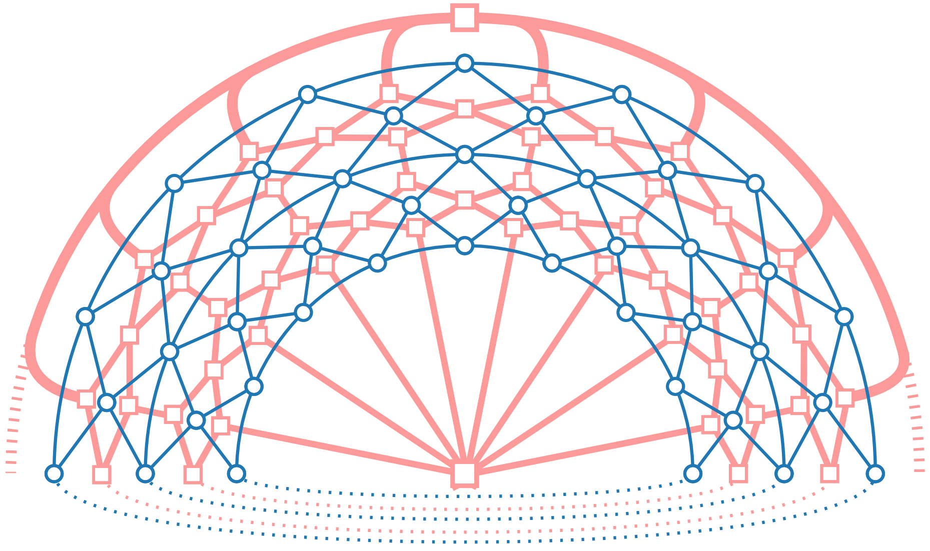

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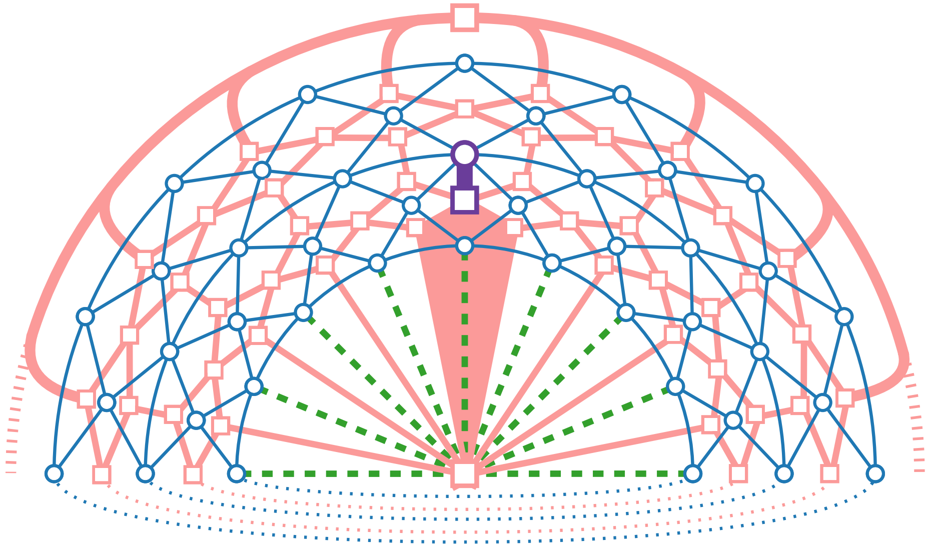
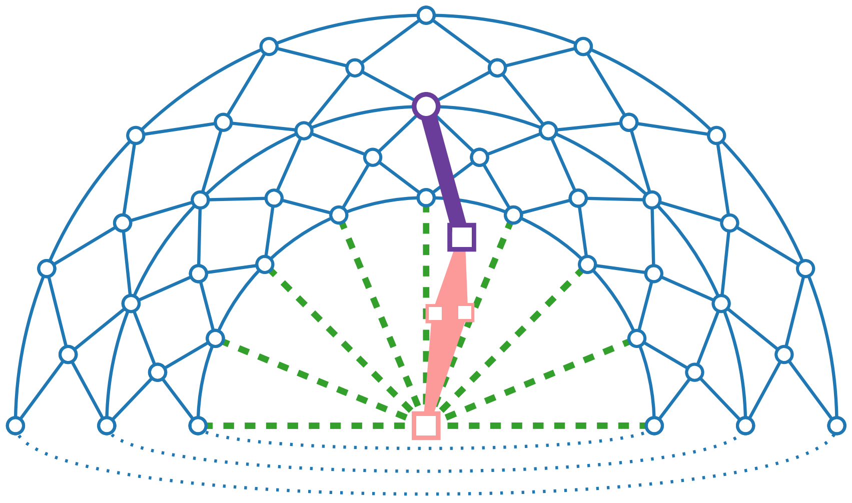
For every $\ell \geq 7$, there is a 1-planar graph G with $n = 11\ell + 2$ vertices such that $\mathrm{cr}(G) = 2$ and $\mathrm{cr}_{1\text{-pl}}(G) = n - 2$.

Crossing ratio
$\rho_{1\text{-pl}}(n) = (n-2)/2$



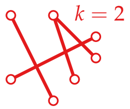
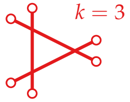
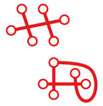
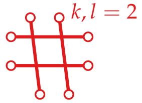
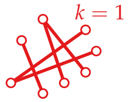
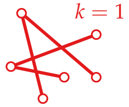
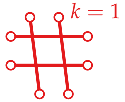
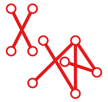
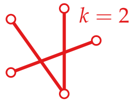
$\mathrm{cr}_{1\text{-pl}}(G) = n - 2$



$\mathrm{cr}(G) = 2$

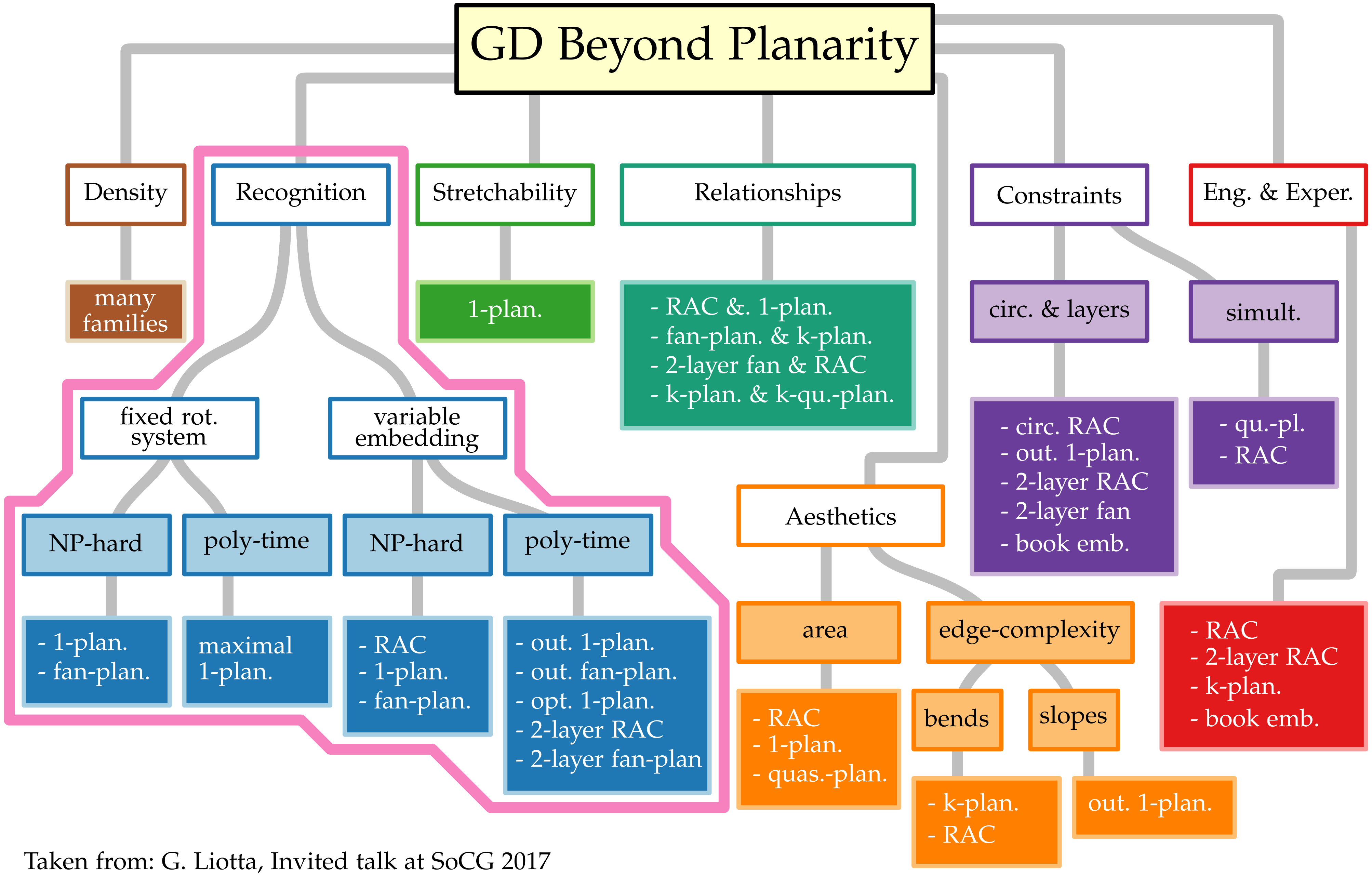
Crossing Ratios

Table from "Crossing Numbers of Beyond-Planar Graphs Revisited"
[van Beusekom, Parada & Speckmann 2021]

Family	Forbidden Configurations		Lower	Upper
k-planar	An edge crossed more than k times		$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(k\sqrt{kn})}$
k-quasi-planar	k pairwise crossing edges		$\Omega(n/k^3)$	$f(k)n^2 \log^2 n$
Fan-planar	Two independent edges crossing a third or two adjacent edges crossing another edge from different "side"		$\Omega(n)$	$O(n^2)$
(k,l)-grid-free	Set of k edges such that each edge crosses each edge from a set of l edges.		$\Omega\left(\frac{n}{kl(k+l)}\right)$	$g(k,l)n^2$
k-gap-planar	More than k crossings mapped to an edge in an optimal mapping		$\boldsymbol{\Omega(n/k^3)}$	$\boldsymbol{O(k\sqrt{kn})}$
Skewness-k	Set of crossings not covered by at most k edges		$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(kn+k^2)}$
k-apex	Set of crossings not covered by at most k vertices		$\Omega(n/k)$	$O(k^2n^2+k^4)$
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge		$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$
k-fan-crossing-free	An edge that crosses k adjacent edges		$\boldsymbol{\Omega(n^2/k^3)}$	$\boldsymbol{O(k^2n^2)}$
Straight-line RAC	Two edges crossing at an angle $< \frac{\pi}{2}$		$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$

Part III: Recognition

GD Beyond Planarity: a Taxonomy



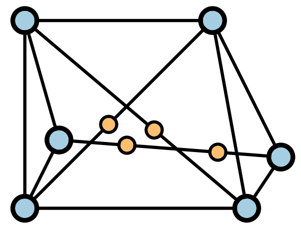
Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

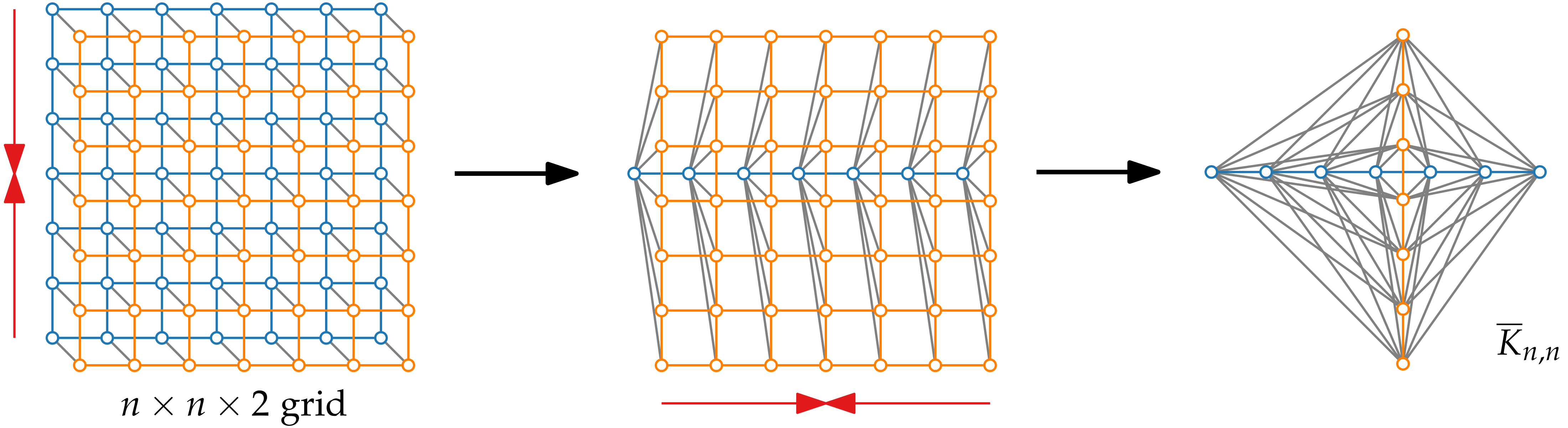
Minors of 1-Planar Graphs

Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$ neither K_5 nor $K_{3,3}$ minor of G

Theorem. [Chen & Kouno 2005]
The class of 1-planar graphs is not closed under edge contraction.

For every graph there is a 1-planar subdivision.



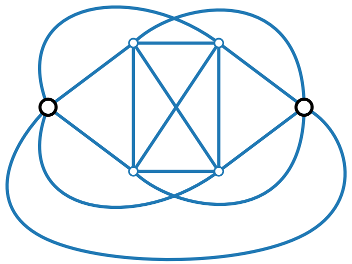


Theorem. [Korzhik & Mohar 2013]
For any n, there exist $\Omega(2^n)$ distinct graphs that are not 1-planar but all their proper subgraphs are 1-planar.

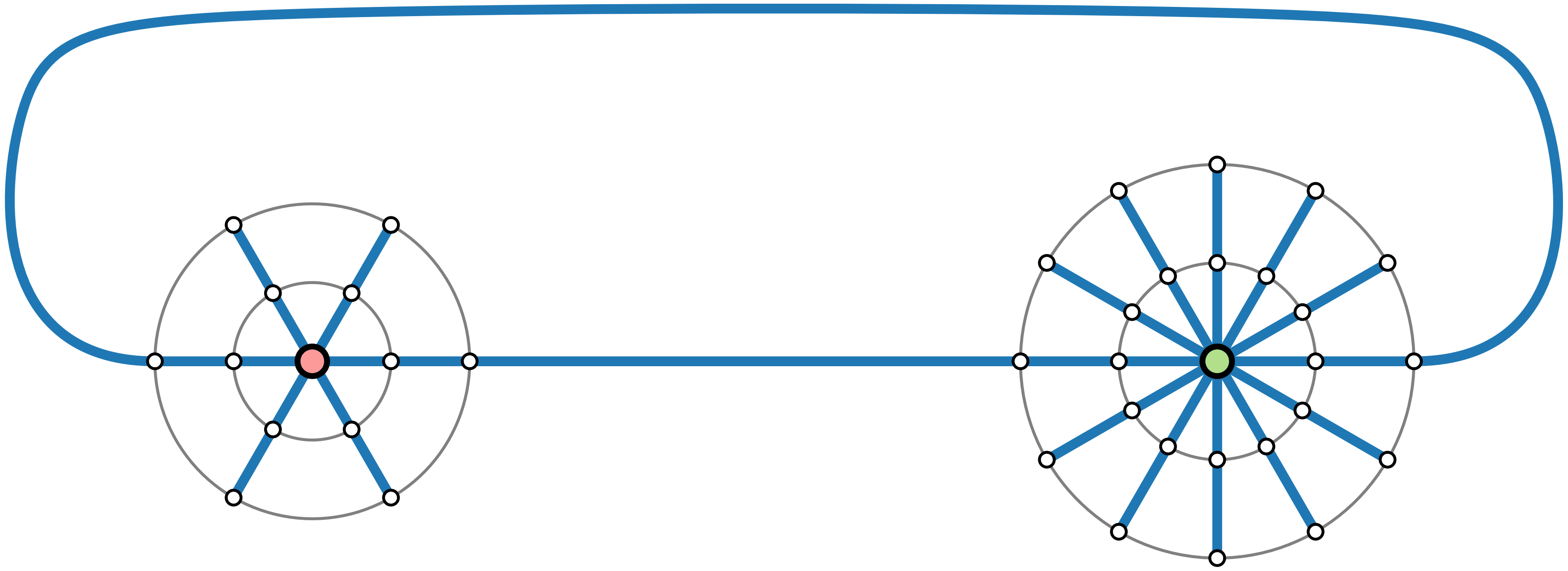
Recognition of 1-Planar Graphs

Theorem. [Grogoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

Proof.
Reduction from 3-Partition.



Only 1-planar embedding of K_6

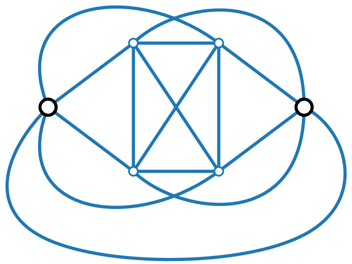


Recognition of 1-Planar Graphs

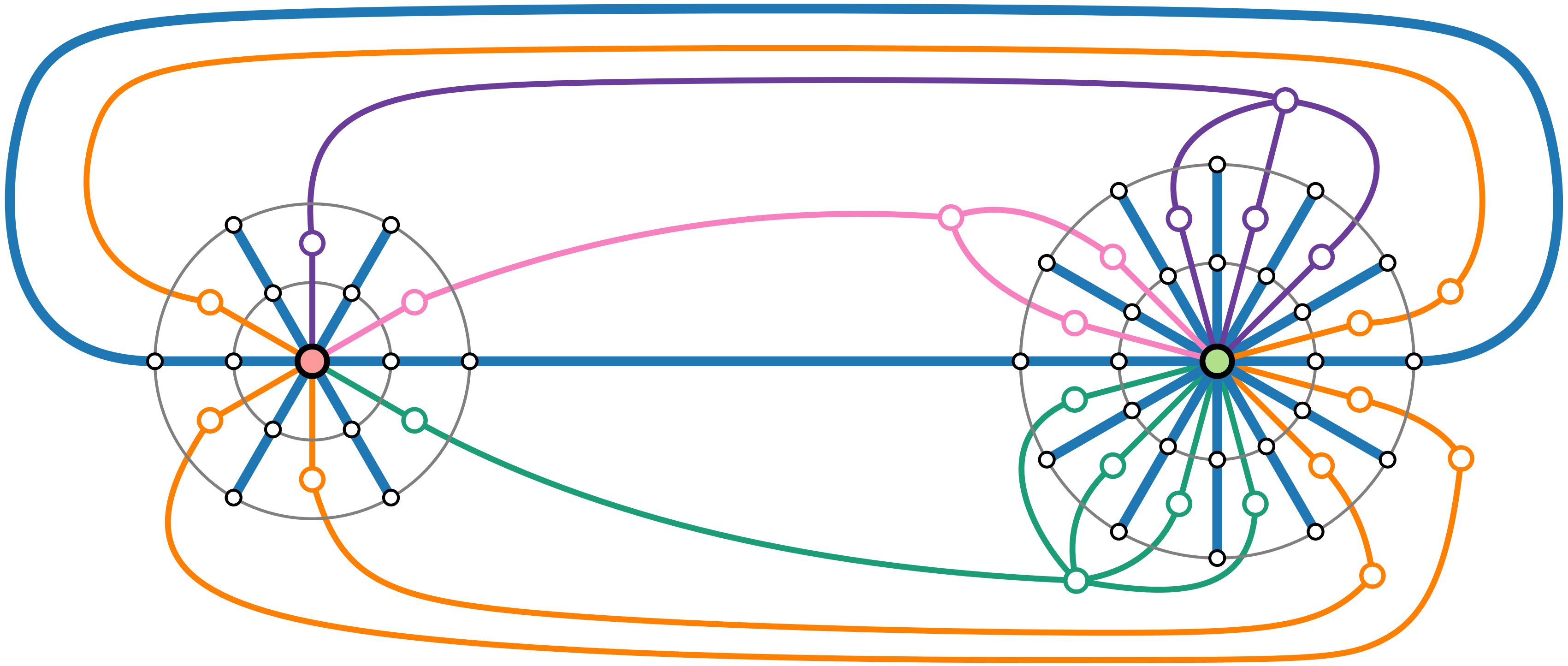
Theorem. [Grogoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

Proof.
Reduction from 3-Partition.

$$A = \{\overbrace{1, 3, 2}^{6}, \overbrace{4, 1, 1}^{6}\}$$



Only 1-planar embedding of K_6



Recognition of 1-Planar Graphs

Theorem. [Grogoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

Theorem. [Cabello & Mohar 2013]
Testing 1-planarity is NP-complete, even for almost planar graphs, i.e., planar graphs plus one edge.

Theorem. [Bannister, Cabello & Eppstein 2018]
Testing 1-planarity is NP-complete, even for graphs of bounded bandwidth (pathwidth, treewidth).

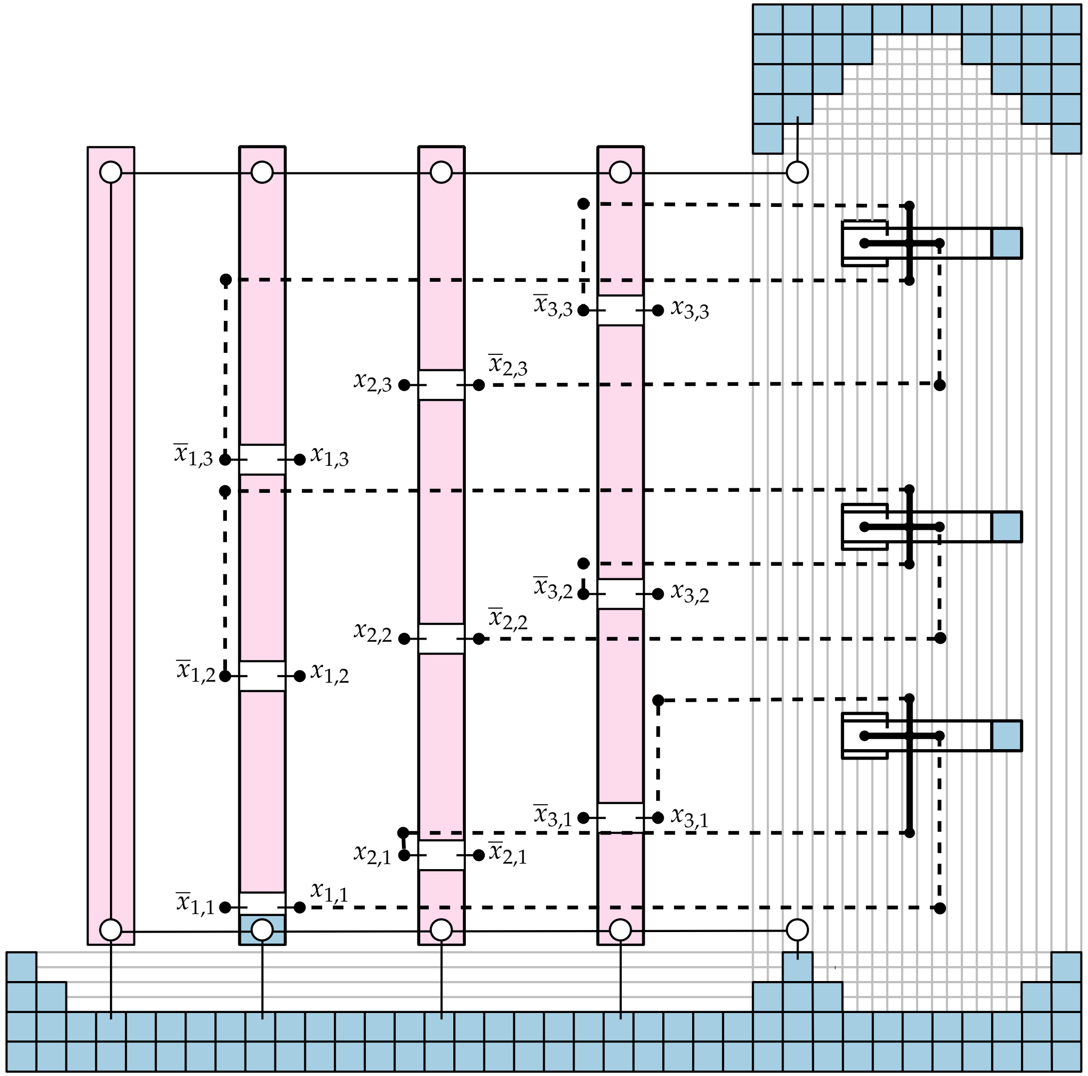
Theorem. [Auer, Brandenburg, Gleißner & Reislhuber 2015]
Testing 1-planarity is NP-complete, even for 3-connected graphs with a fixed rotation system.

Recognition of RAC Graphs

Theorem. [Argyriou, Bekos, Symvonis 2013]
Testing whether a graph is RAC is NP-complete.

Reduction idea (from 3-SAT)

- $x_1 = true$
- $x_2 = false$
- $x_3 = true$



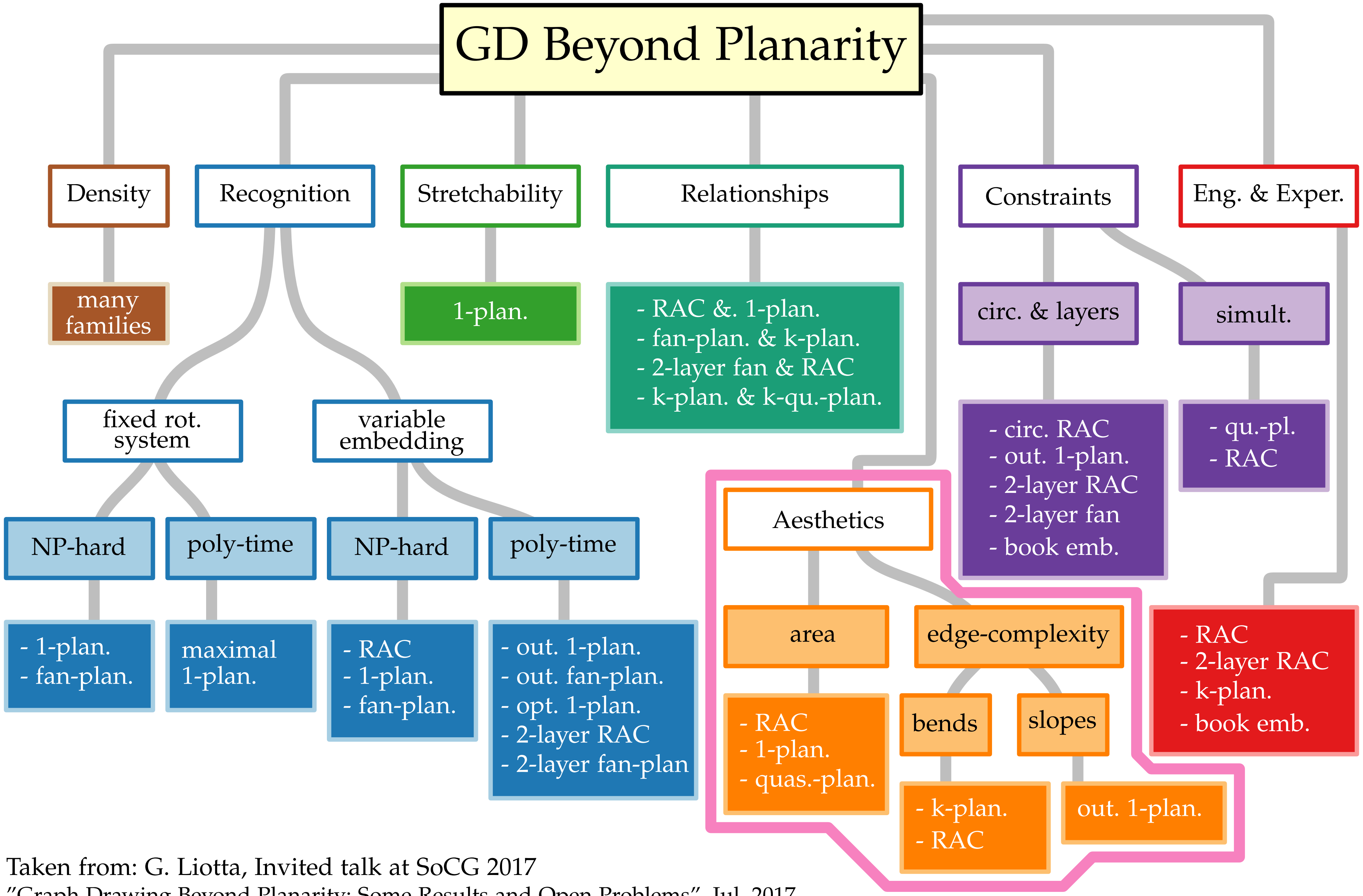
$C_3 = \overline{x}_1 \vee \overline{x}_2 \vee x_3$

$C_2 = \overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3$

$C_1 = x_1 \vee x_2 \vee x_3$

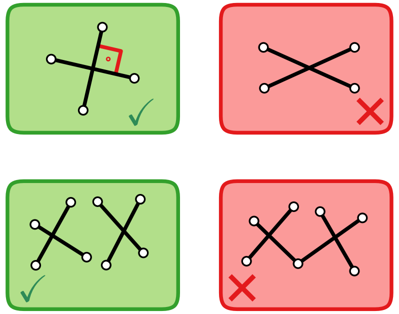
Part IV:
RAC Drawings

GD Beyond Planarity: a Taxonomy

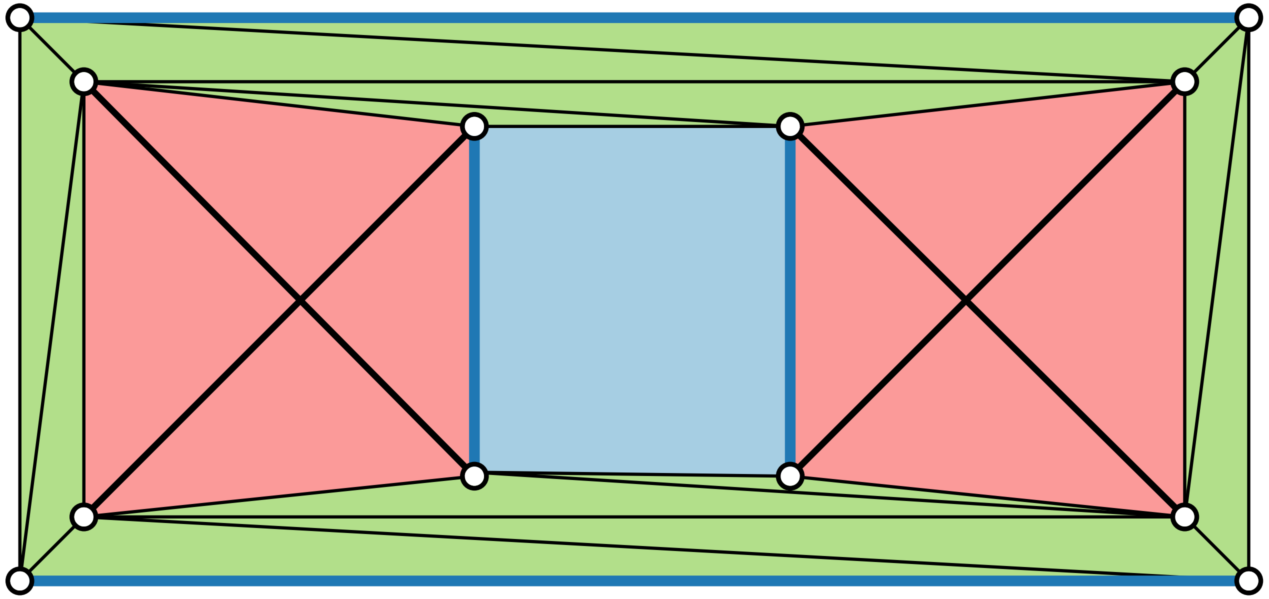


Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

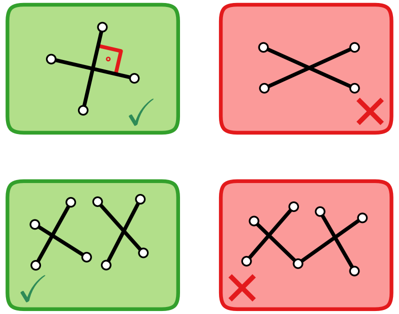
Area of Straight-Line RAC Drawings



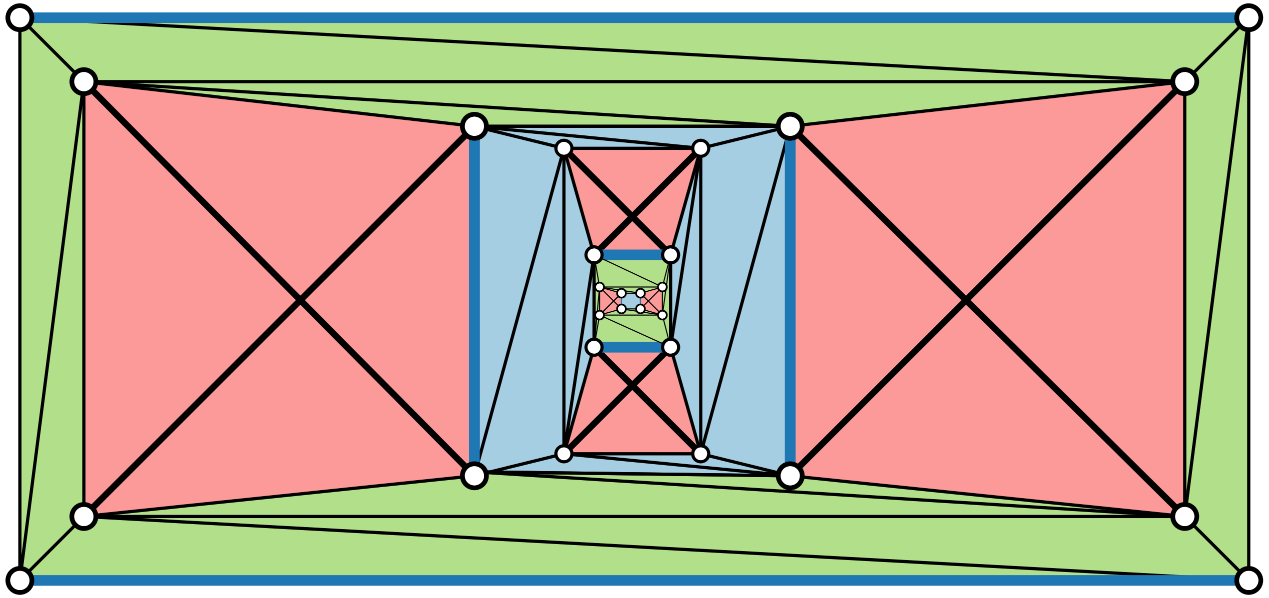
Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
IC-planar straight-line RAC drawings may require exponential area.



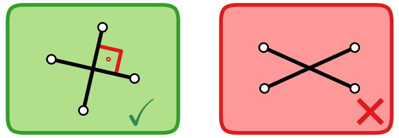
Area of Straight-Line RAC Drawings

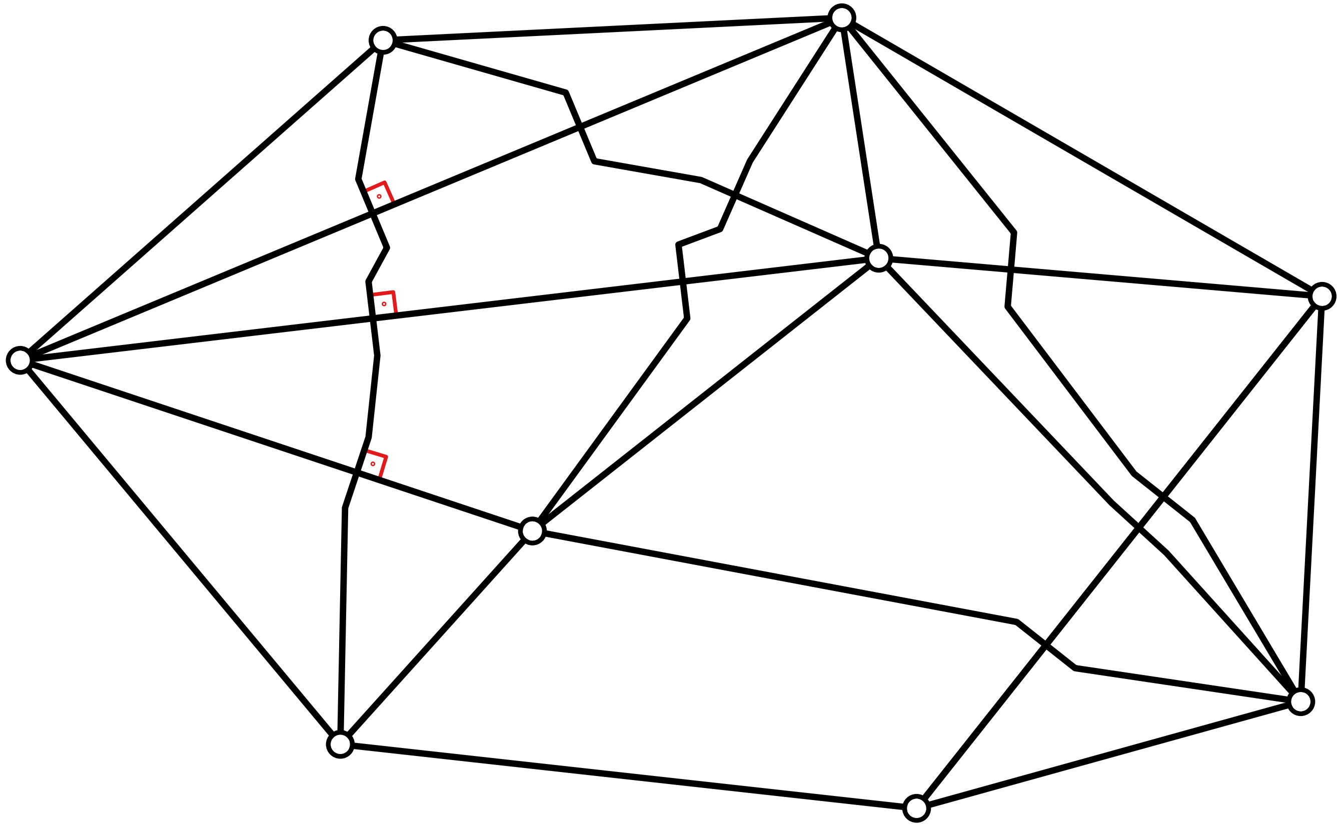


Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
IC-planar straight-line RAC drawings may require exponential area.



RAC Drawings With Enough Bends



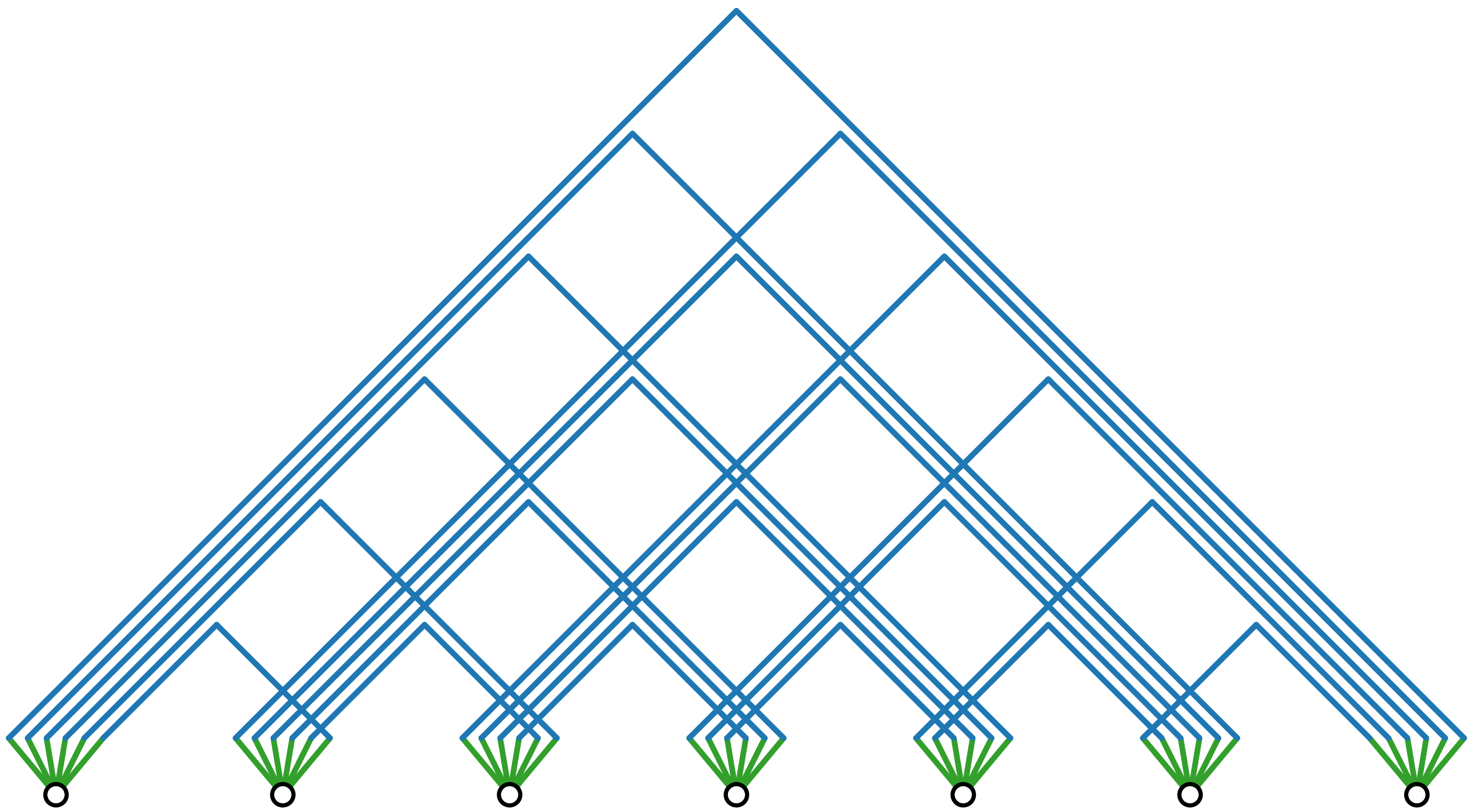


Every graph admits a RAC drawing . . .
. . . if we use enough bends.

How many do we need at most in total or per edge?

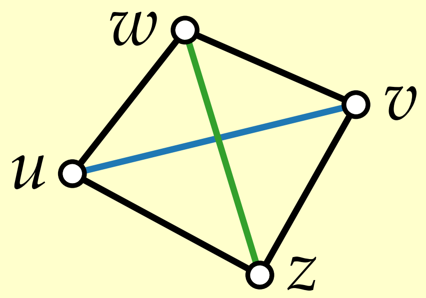
3-Bend RAC Drawings

Theorem. [Didimo, Eades & Liotta 2007]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most 3 bends.

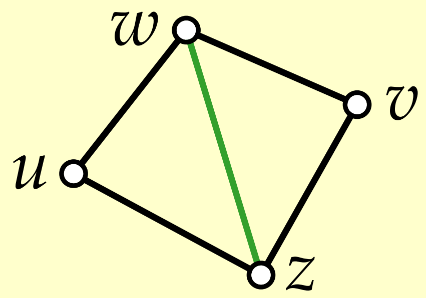


Kite Triangulations

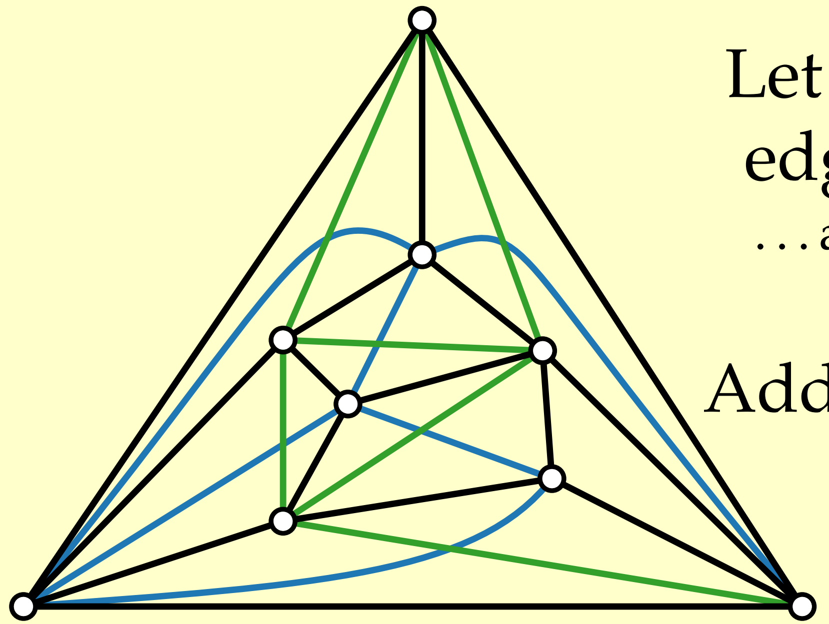
This is a **kite**:



u and v are **opposite** wrt $\{z, w\}$



Let G' be a plane triangulation.



Let $S \subset E(G')$ s.t. no two edges in S on same face.

... and their opposite vertices do not have an edge in $E(G')$.

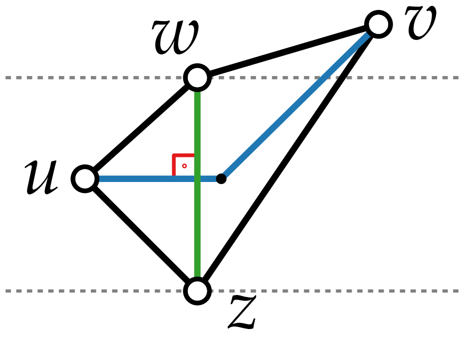
Add edges T for opposite vertices wrt to S.

The resulting graph G is a **kite-triangulation**.

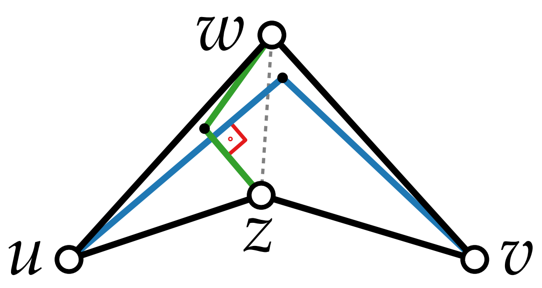
optimal 1-planar $\subset$ kite-triangulation

Theorem. [Angelini et al. '11]
Every kite-triangulation G on n vertices admits a 1-planar 1-bend RAC drawing Γ and Γ can be constructed in $\mathcal{O}(n)$ time.

Proof.
Let G' be the underlying plane triang. of G. Let G'' be G' without S.
Construct straight-line drawing of G''
Fill faces as follows:



strictly convex face



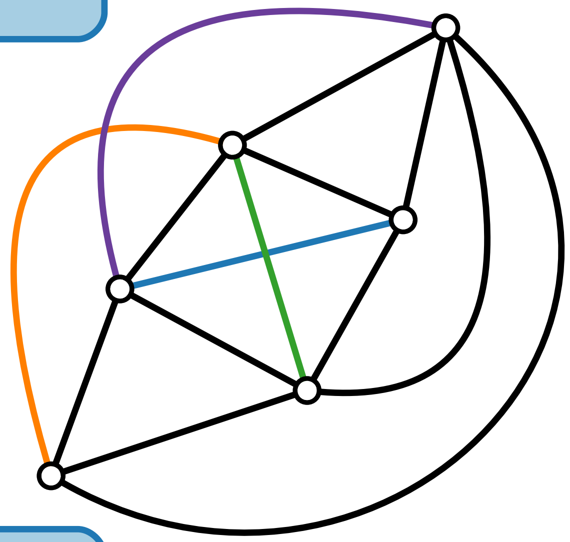
otherwise

Part V:
1-Planar 1-Bend RAC Drawings

1-Planar 1-Bend RAC Drawings

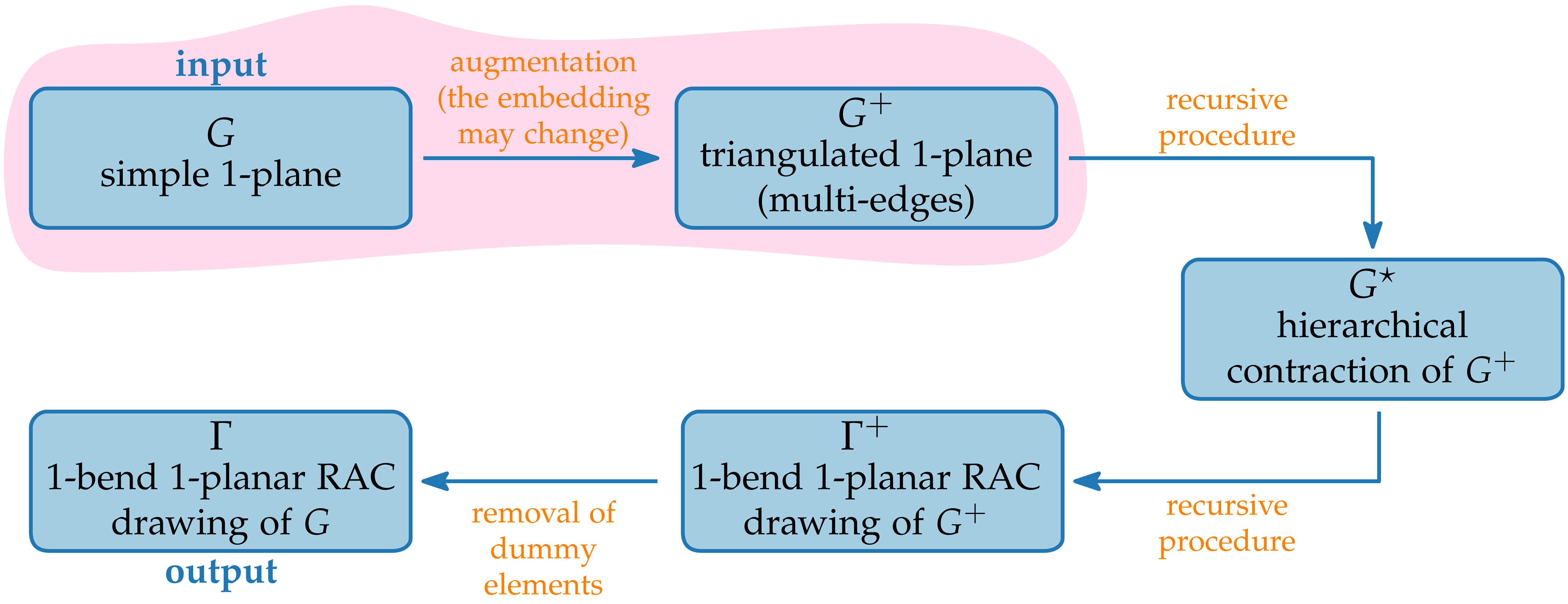
Theorem. [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph G on n vertices admits a 1-planar 1-bend RAC drawing Γ.
Also, if a 1-planar embedding of G is given as part of the input, Γ can be computed in $\mathcal{O}(n)$ time.

Observation.
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of G forms an (empty) kite, except for at most one pair if their crossing point is on the outer face of G.



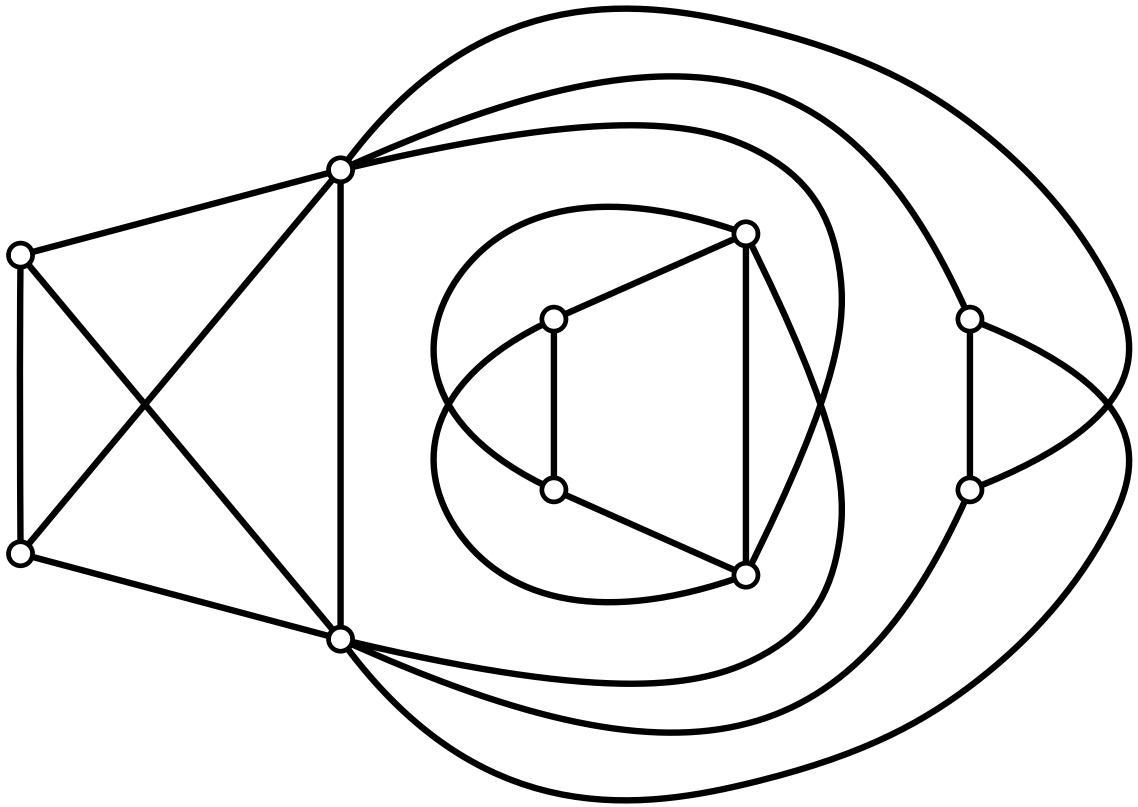
Theorem. [Chiba, Yamanouchi & Nishizeki 1984]
For every planar graph G and convex polygon P, a strictly convex planar straight-line drawing of G where the outer face coincides with P can be computed in $O(n)$ time.

Algorithm Outline



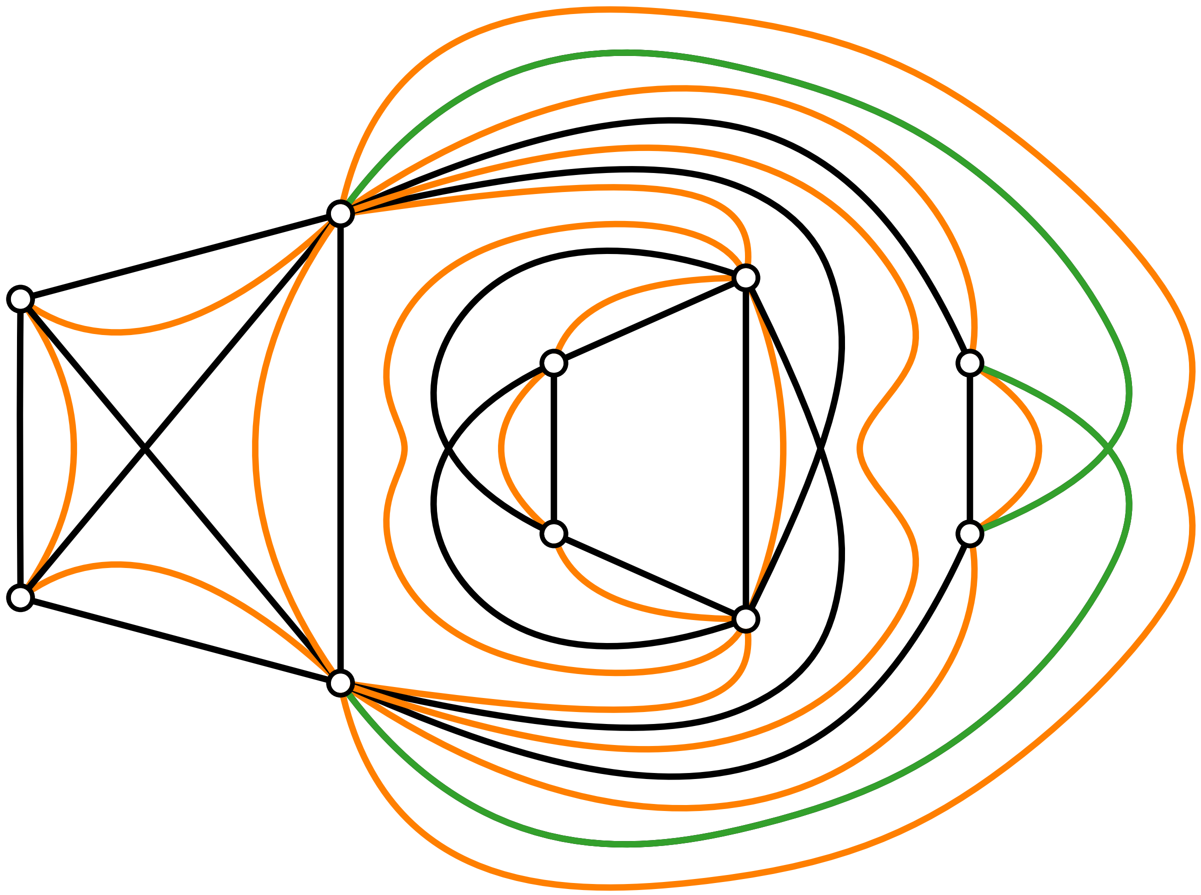
Algorithm Step 1: Augmentation

G
simple 1-plane



Algorithm Step 1: Augmentation

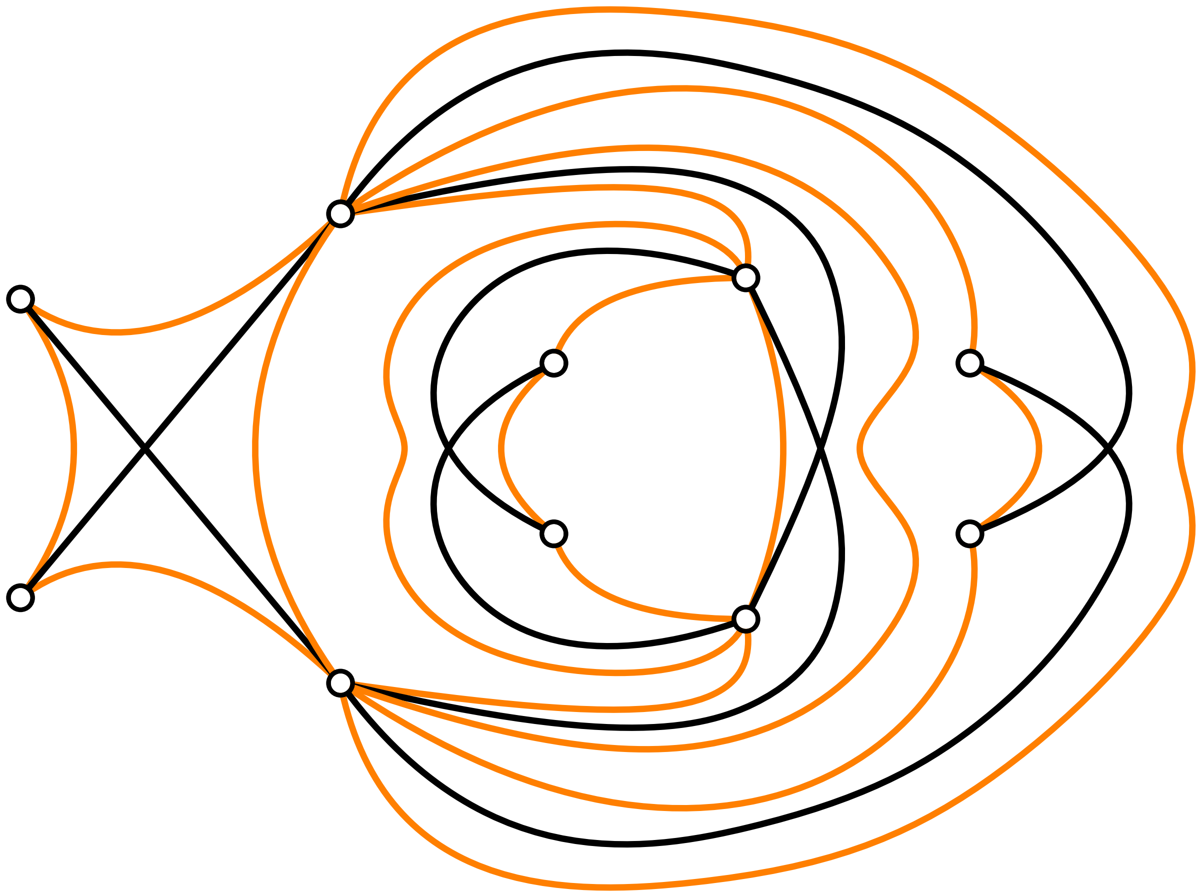
1. For each pair of crossing edges add an enclosing 4-cycle.



Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to *G*.



Algorithm Step 1: Augmentation

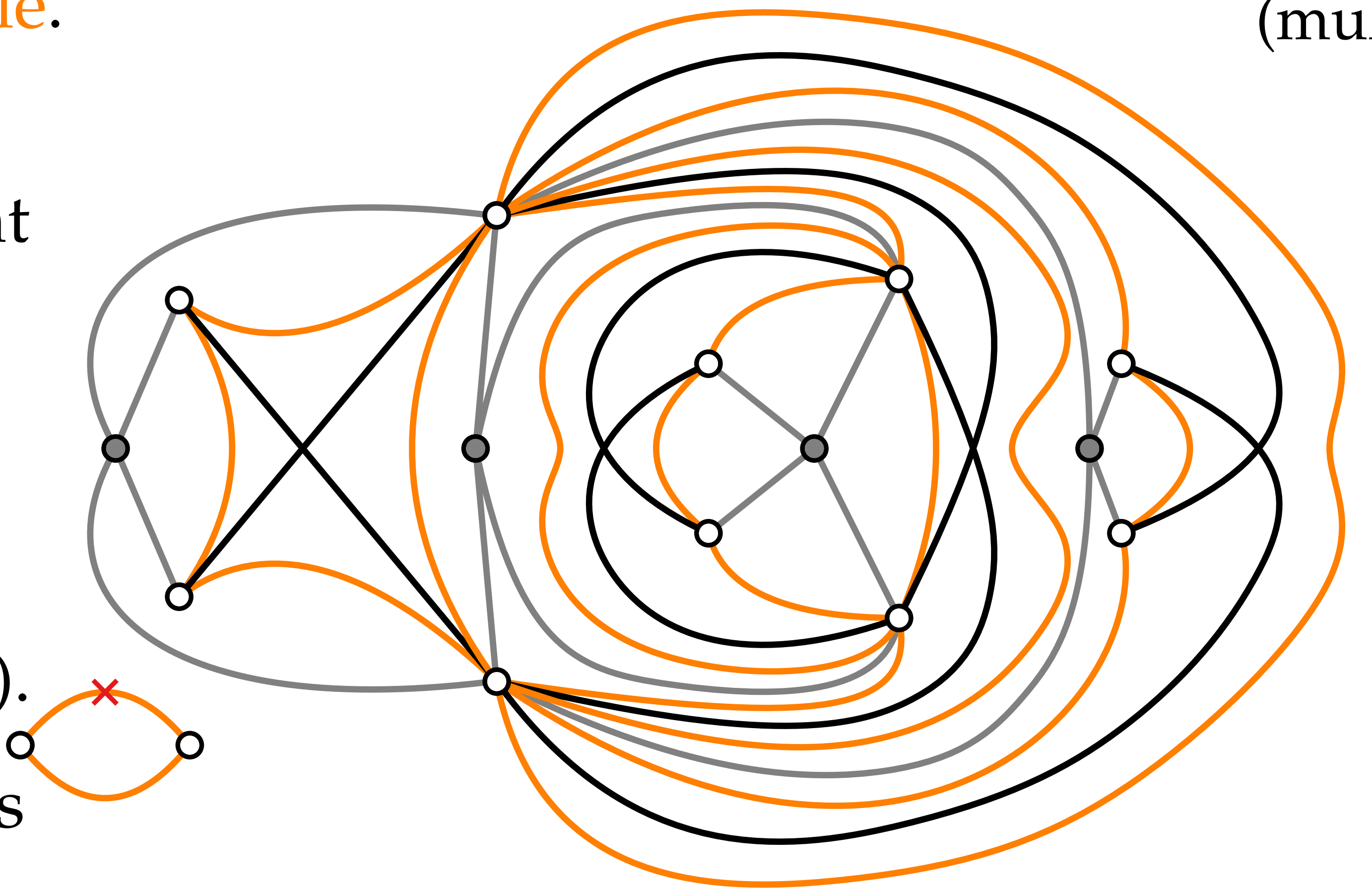
1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to G.

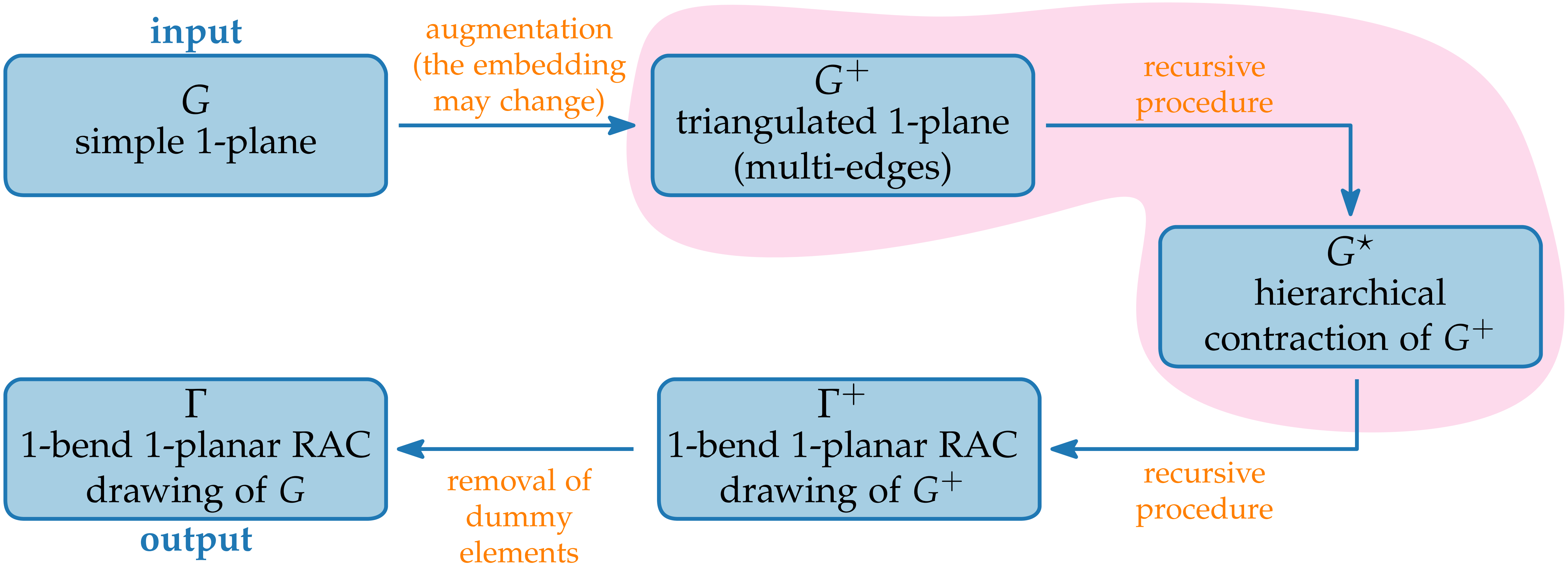
3. Remove one (multiple) edge from each face of degree two (if any).

4. Triangulate faces of degree > 3 by inserting a star inside them.

G^+
triangulated 1-plane
(multi-edges)



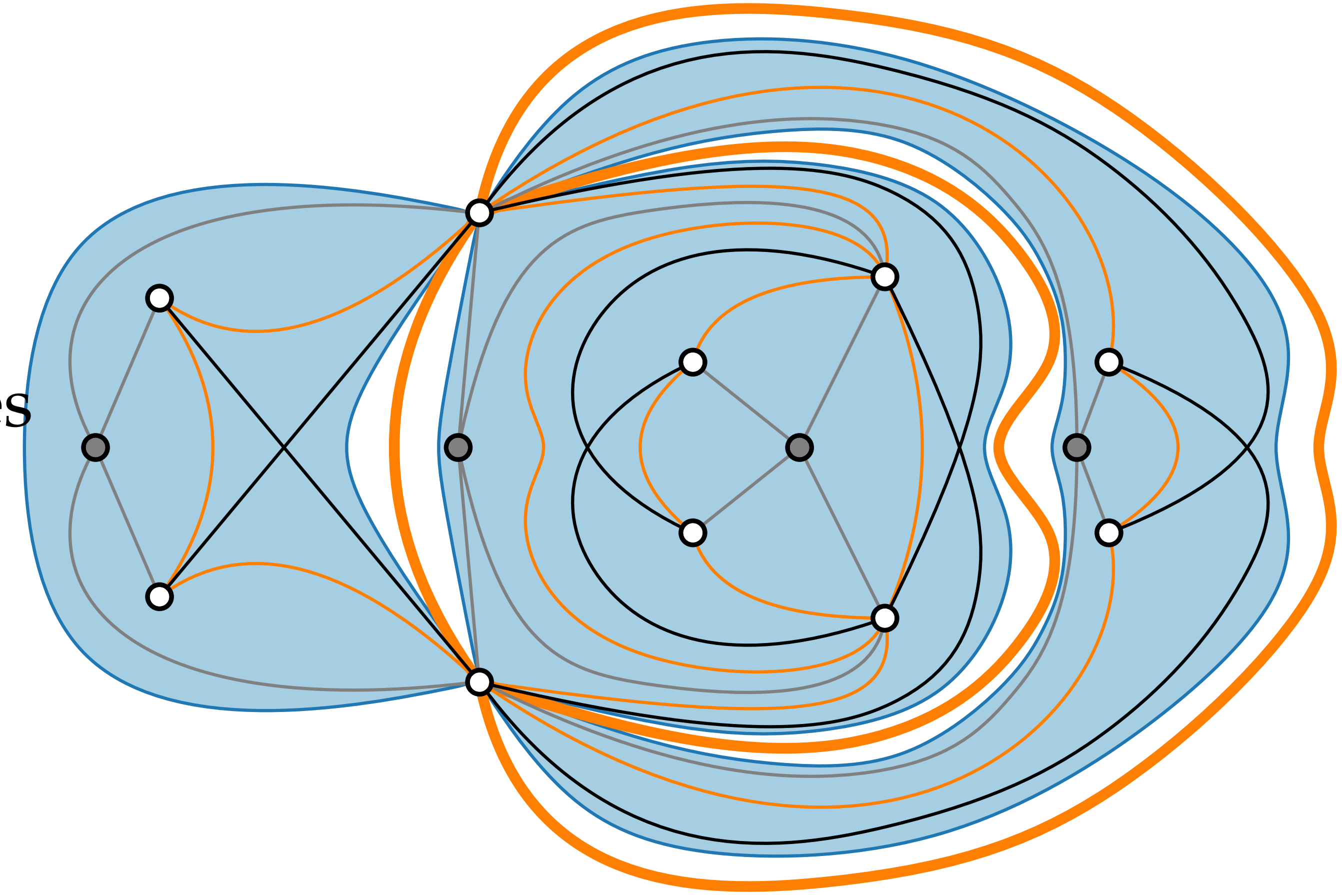
Algorithm Outline



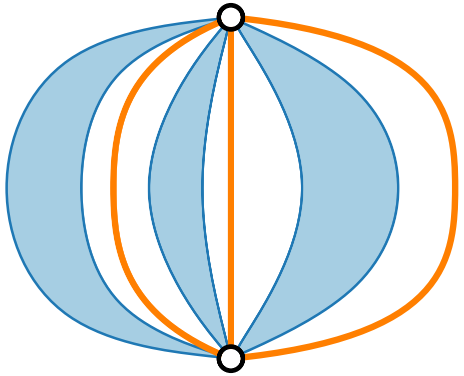
Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



$\Rightarrow$

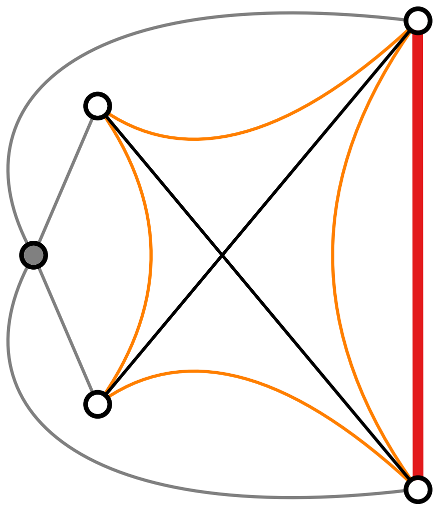


structure of each separation pair

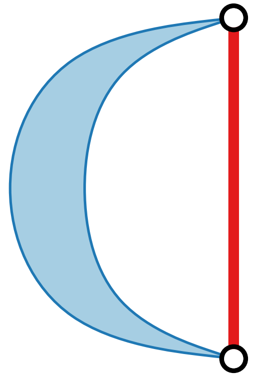
Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



$\Rightarrow$



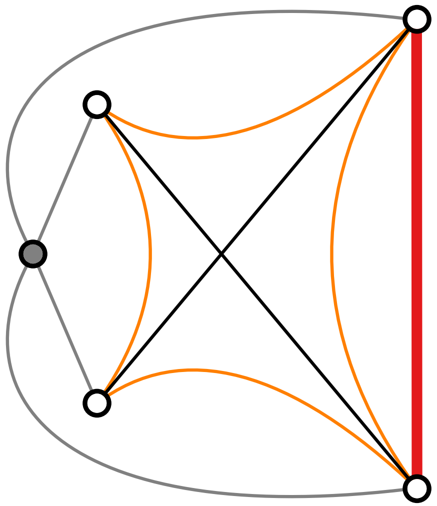
structure of each separation pair

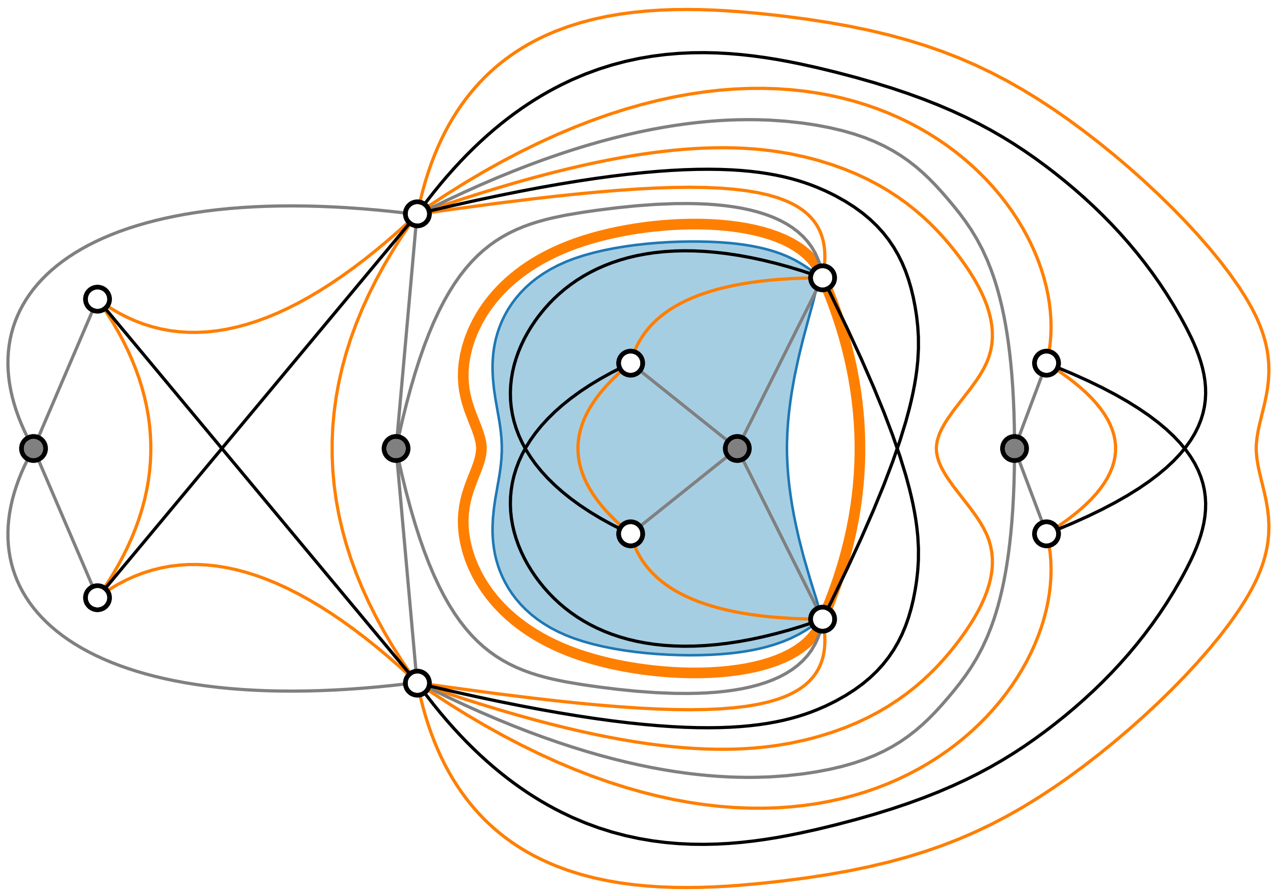
Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

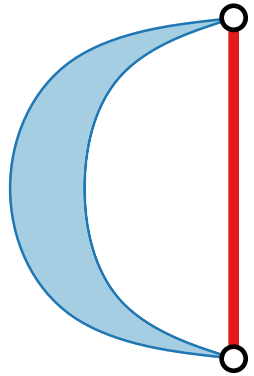
G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites





$\Rightarrow$



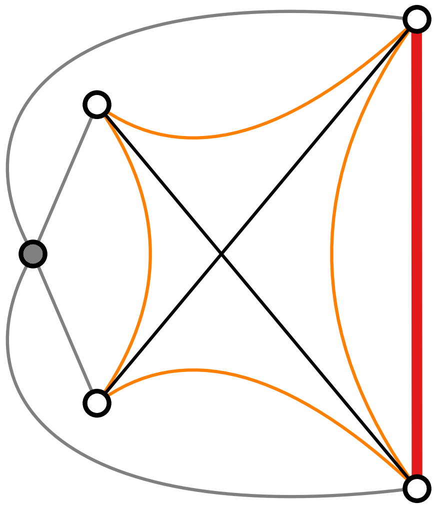
structure of each separation pair

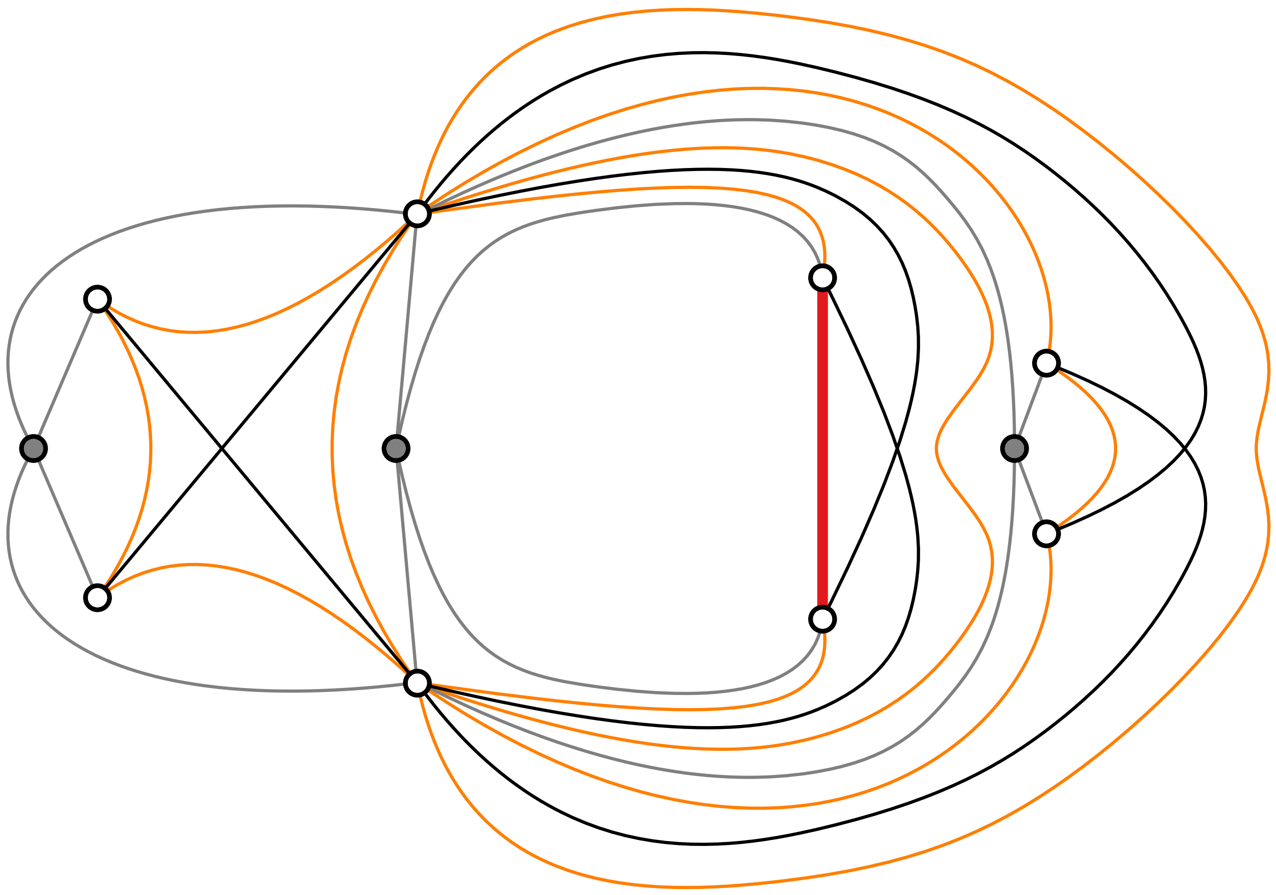
Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

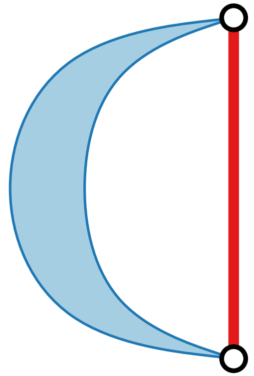
G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites





$\Rightarrow$

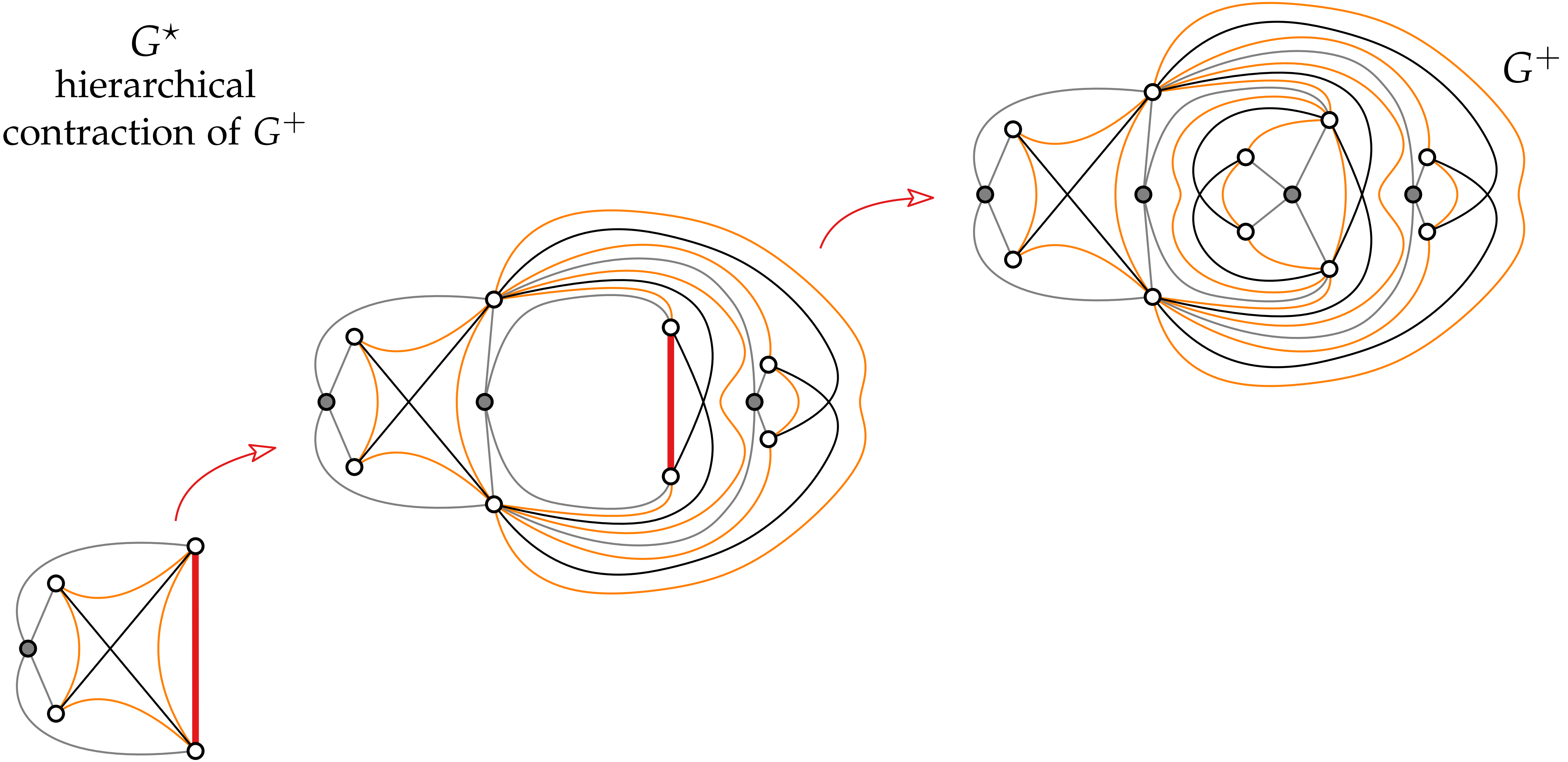


structure of each separation pair

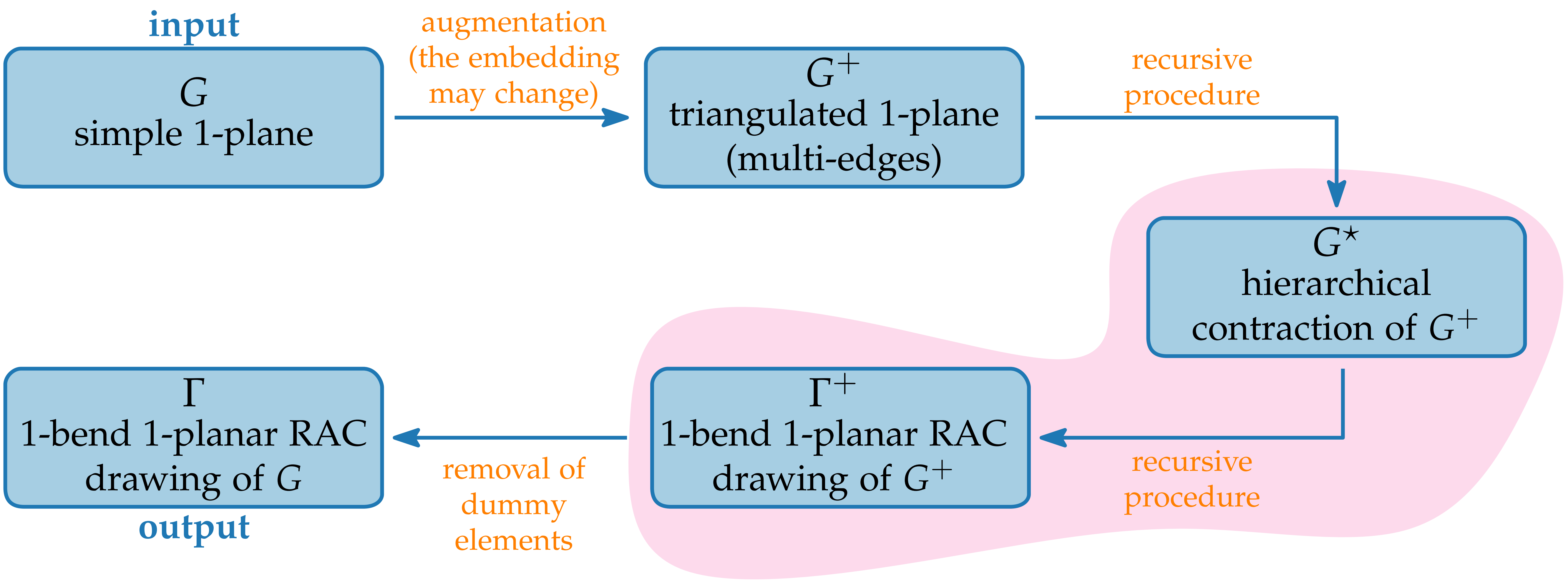
Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

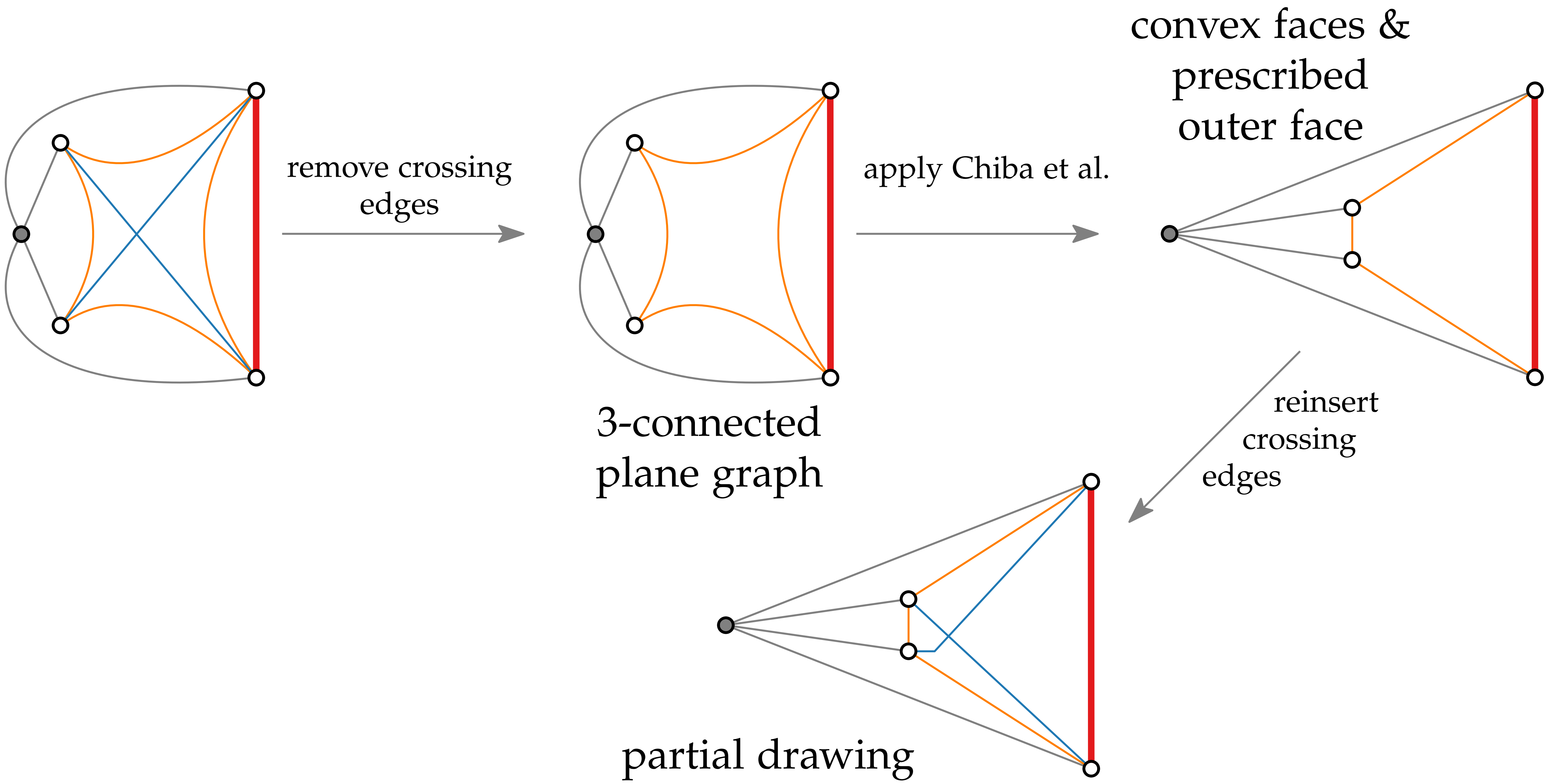
$G^\star$
hierarchical
contraction of G^+



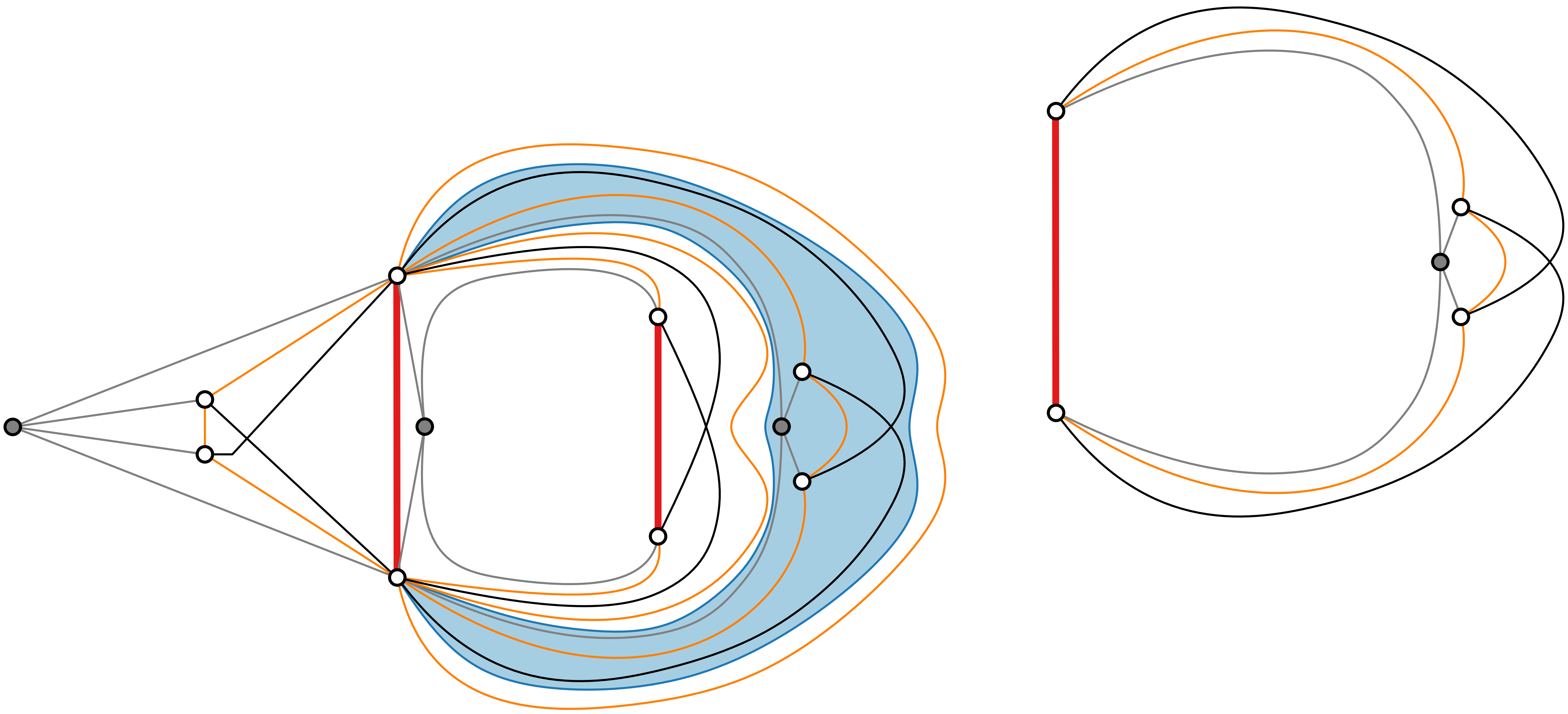
Algorithm Outline



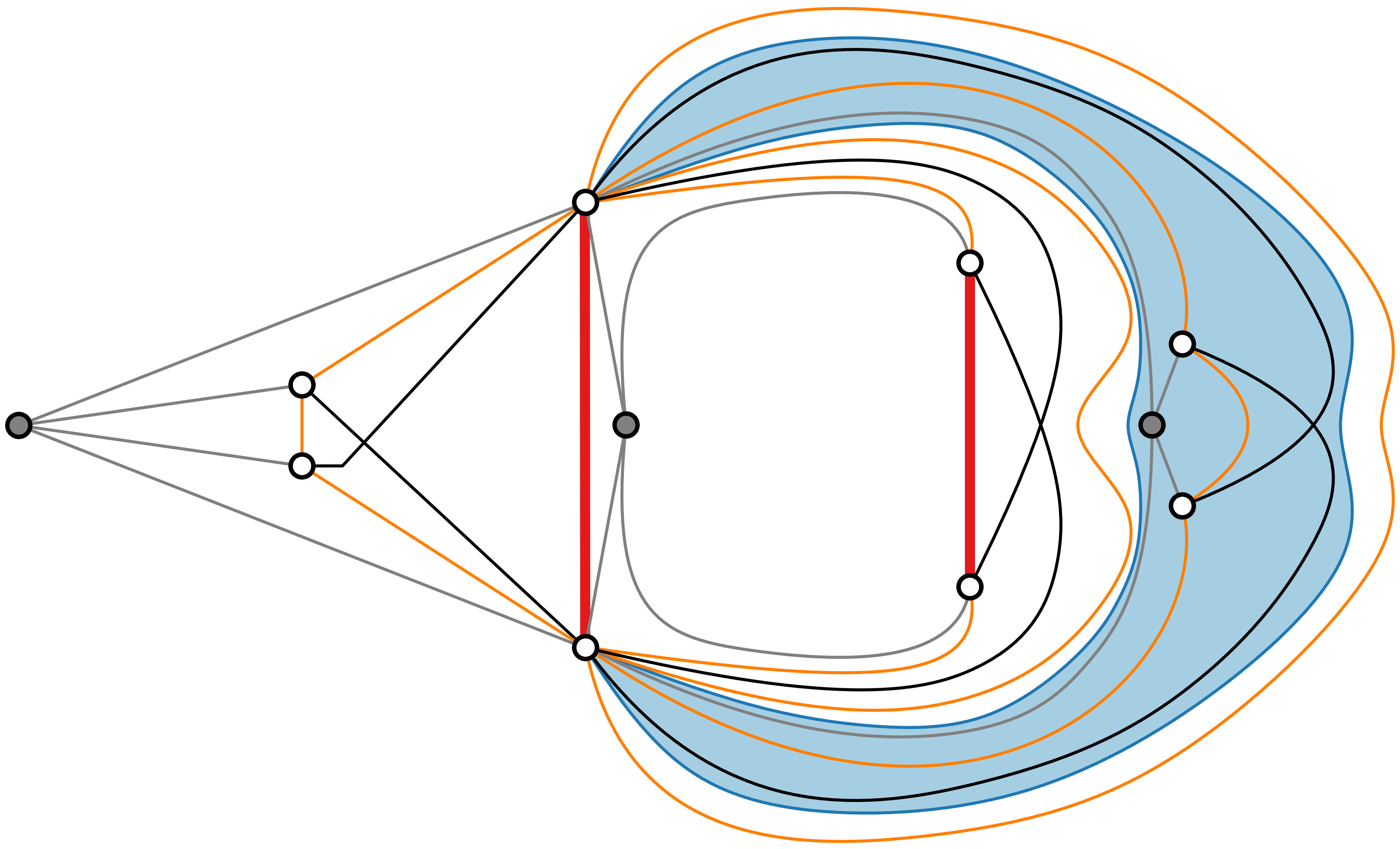
Algorithm Step 3: Drawing Procedure

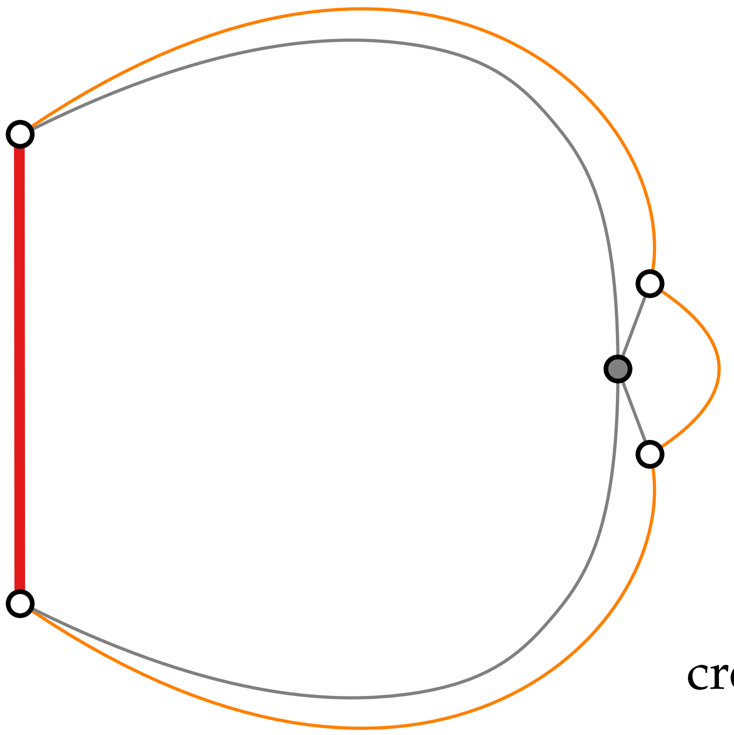


Algorithm Step 3: Drawing Procedure



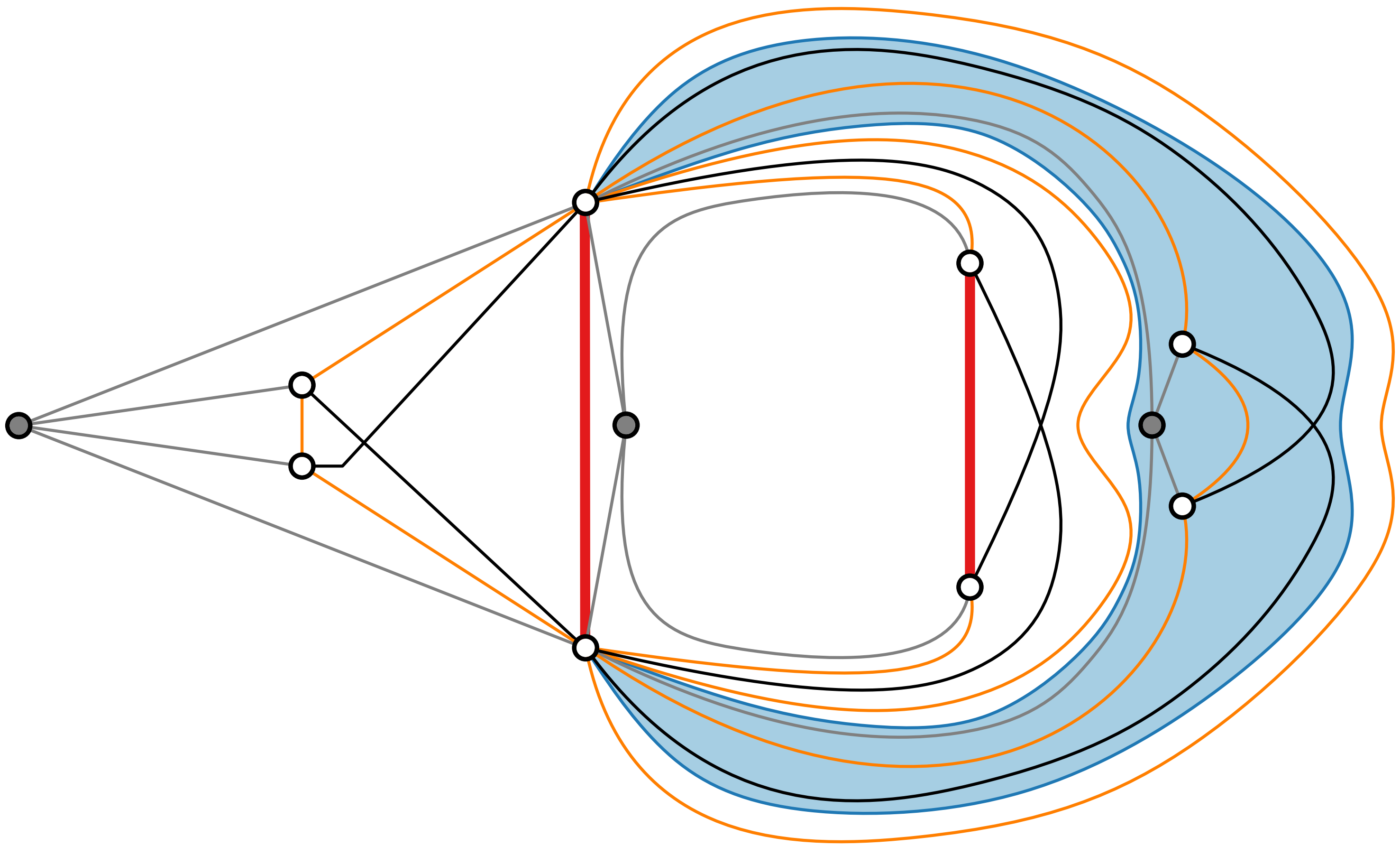
Algorithm Step 3: Drawing Procedure

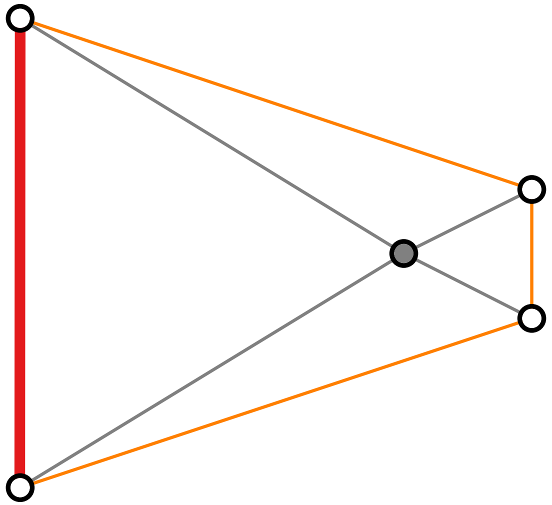




remove
crossing edges

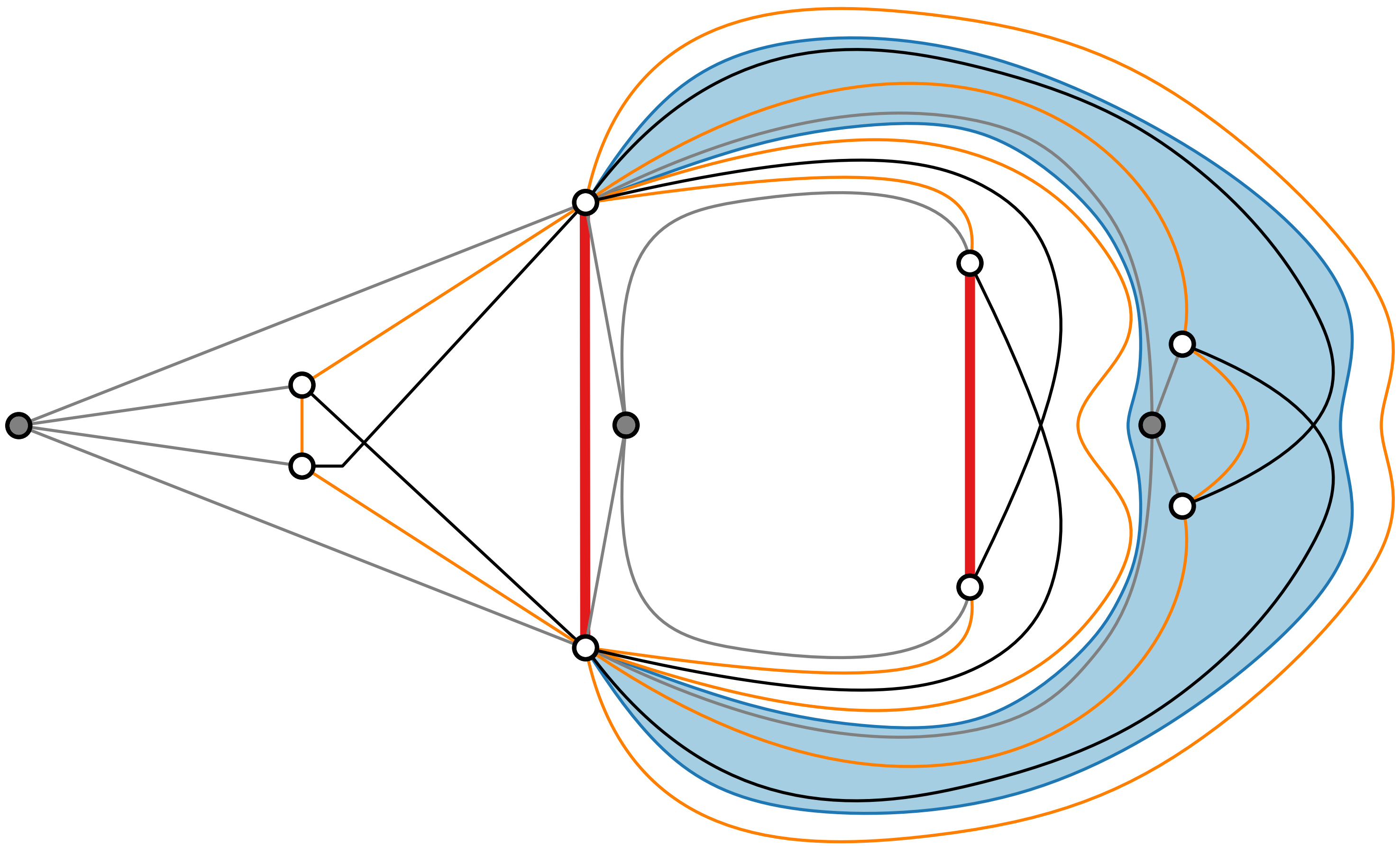
Algorithm Step 3: Drawing Procedure

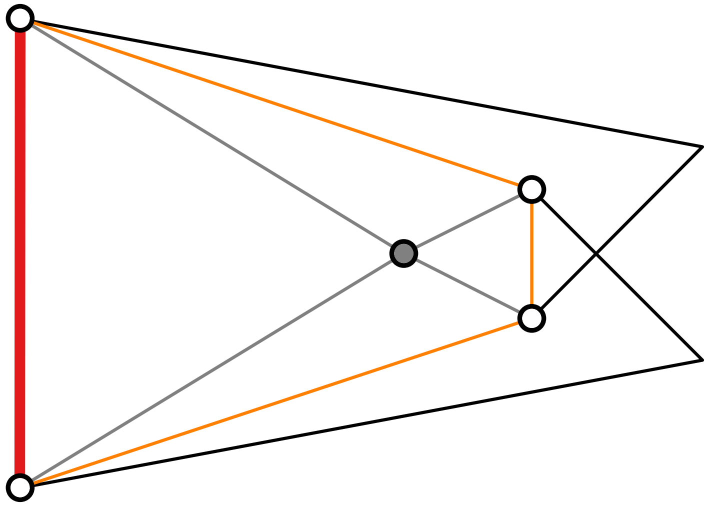




apply Chiba et al.

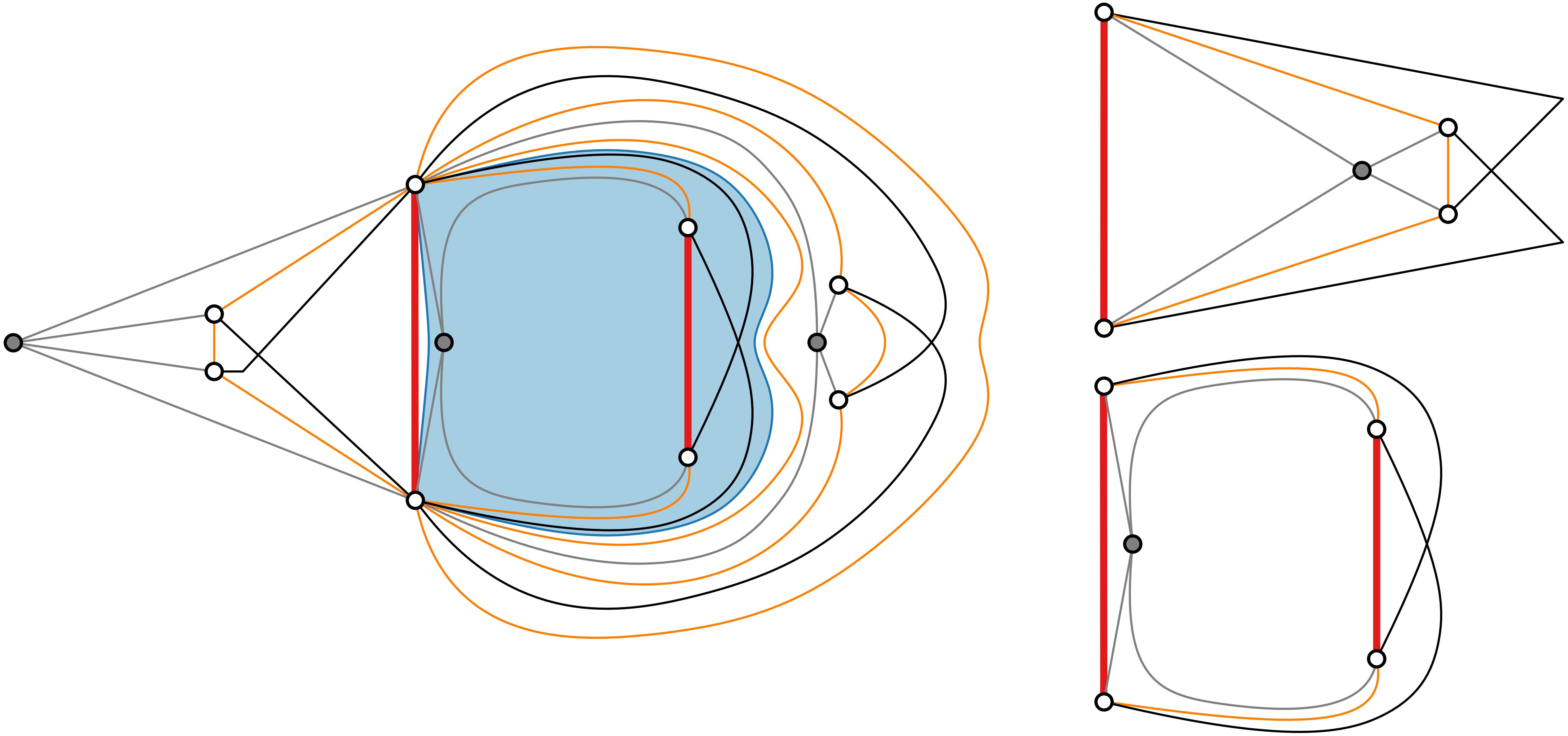
Algorithm Step 3: Drawing Procedure



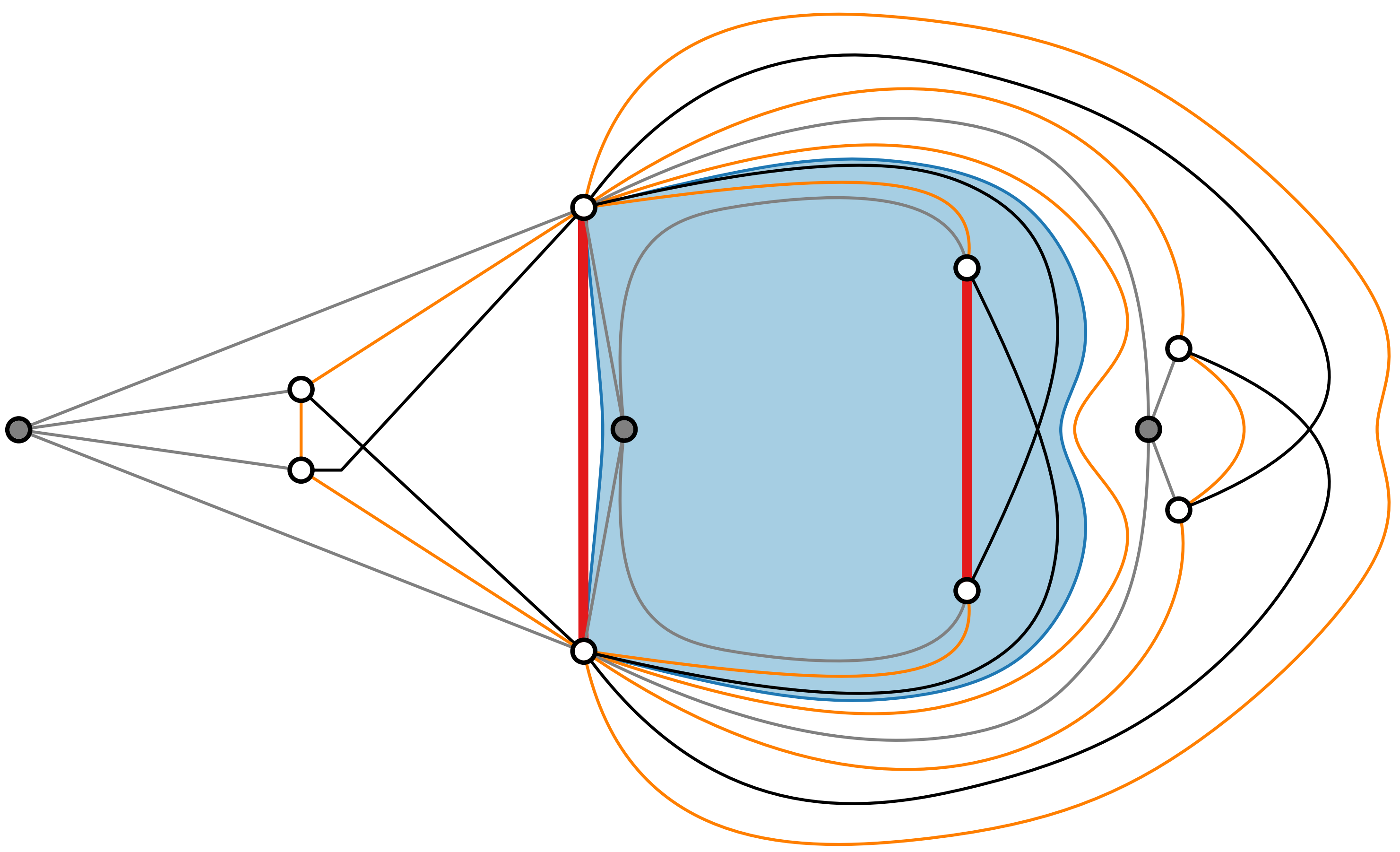


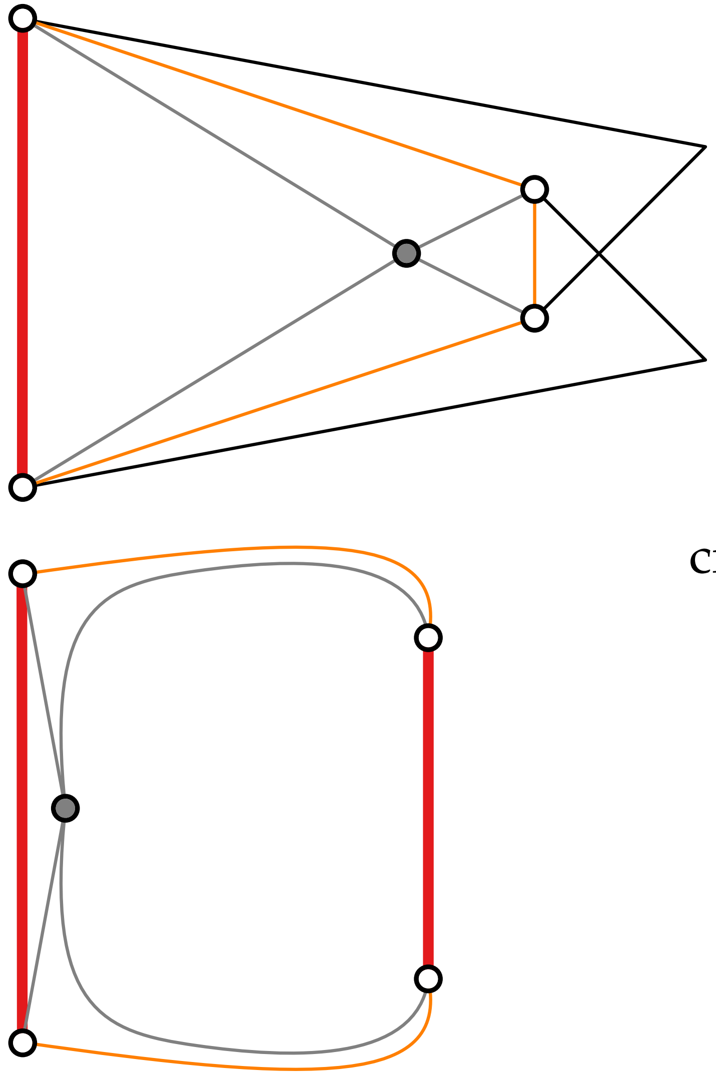
reinsert
crossing edges

Algorithm Step 3: Drawing Procedure



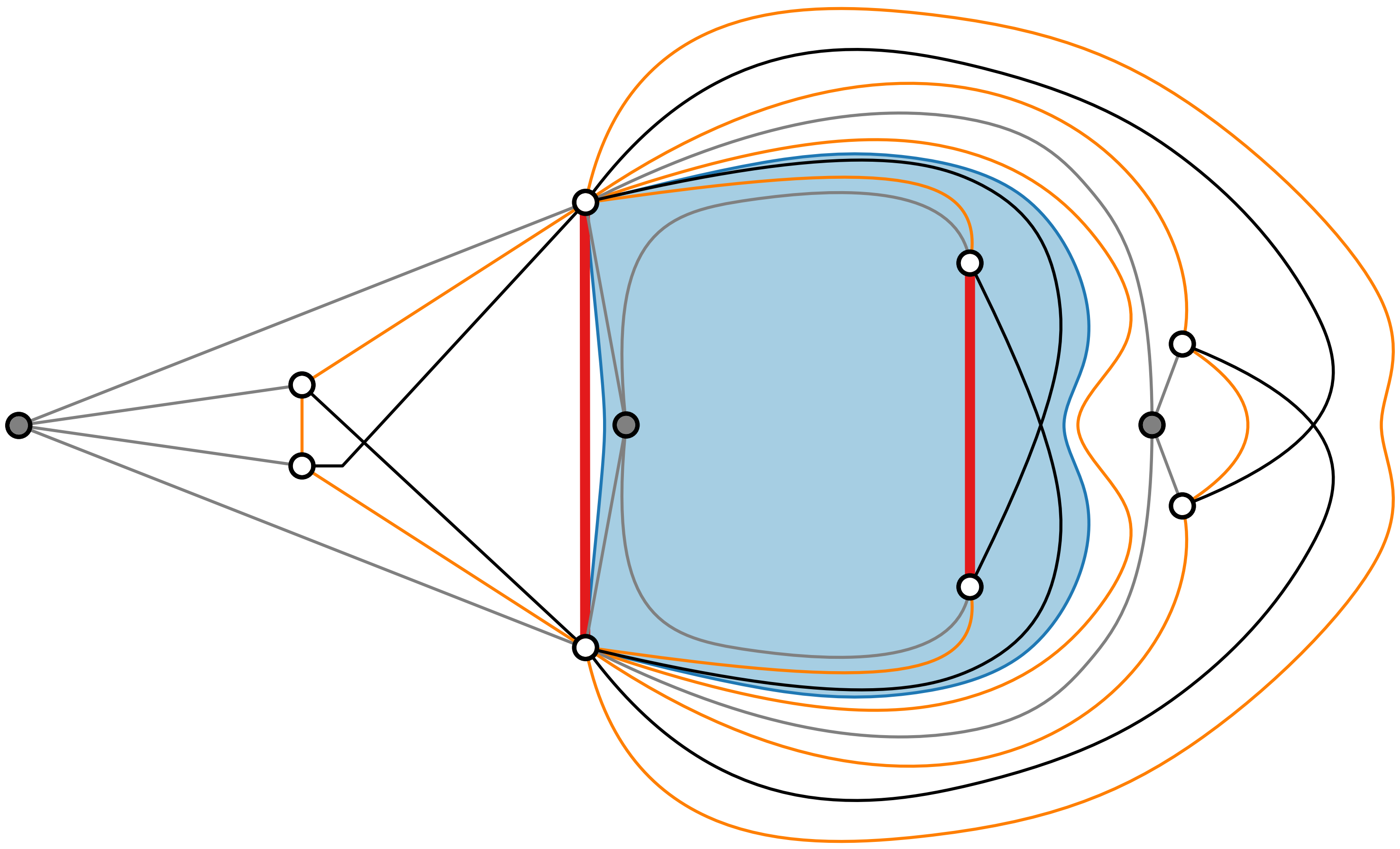
Algorithm Step 3: Drawing Procedure

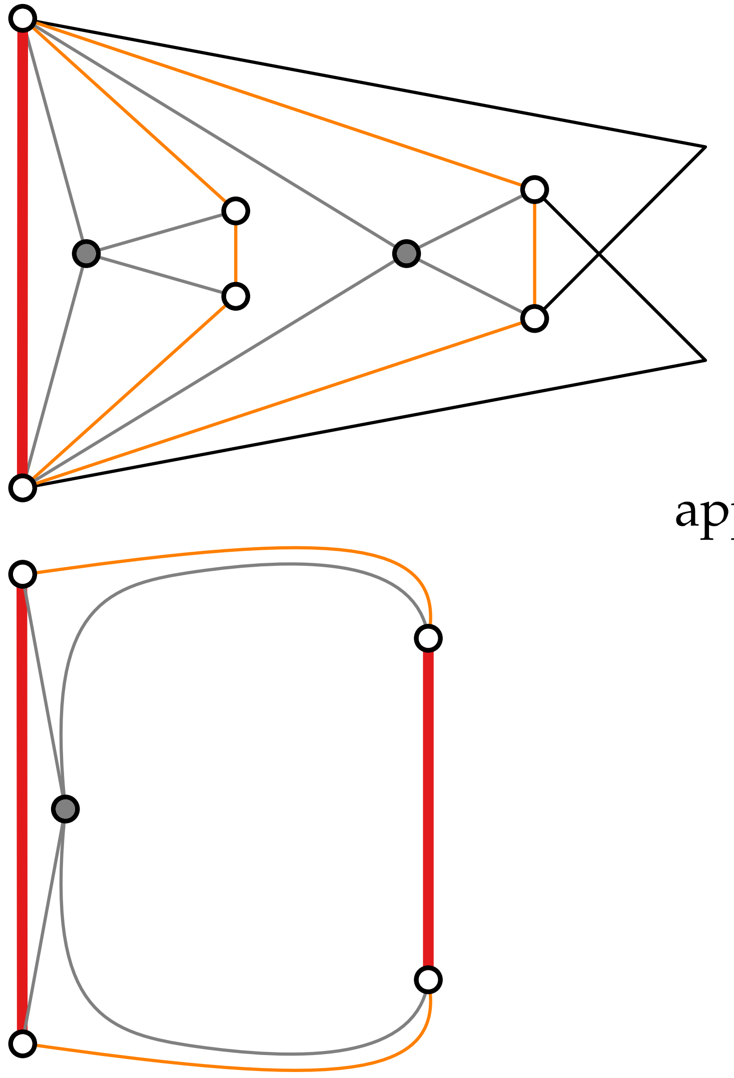




remove
crossing edges

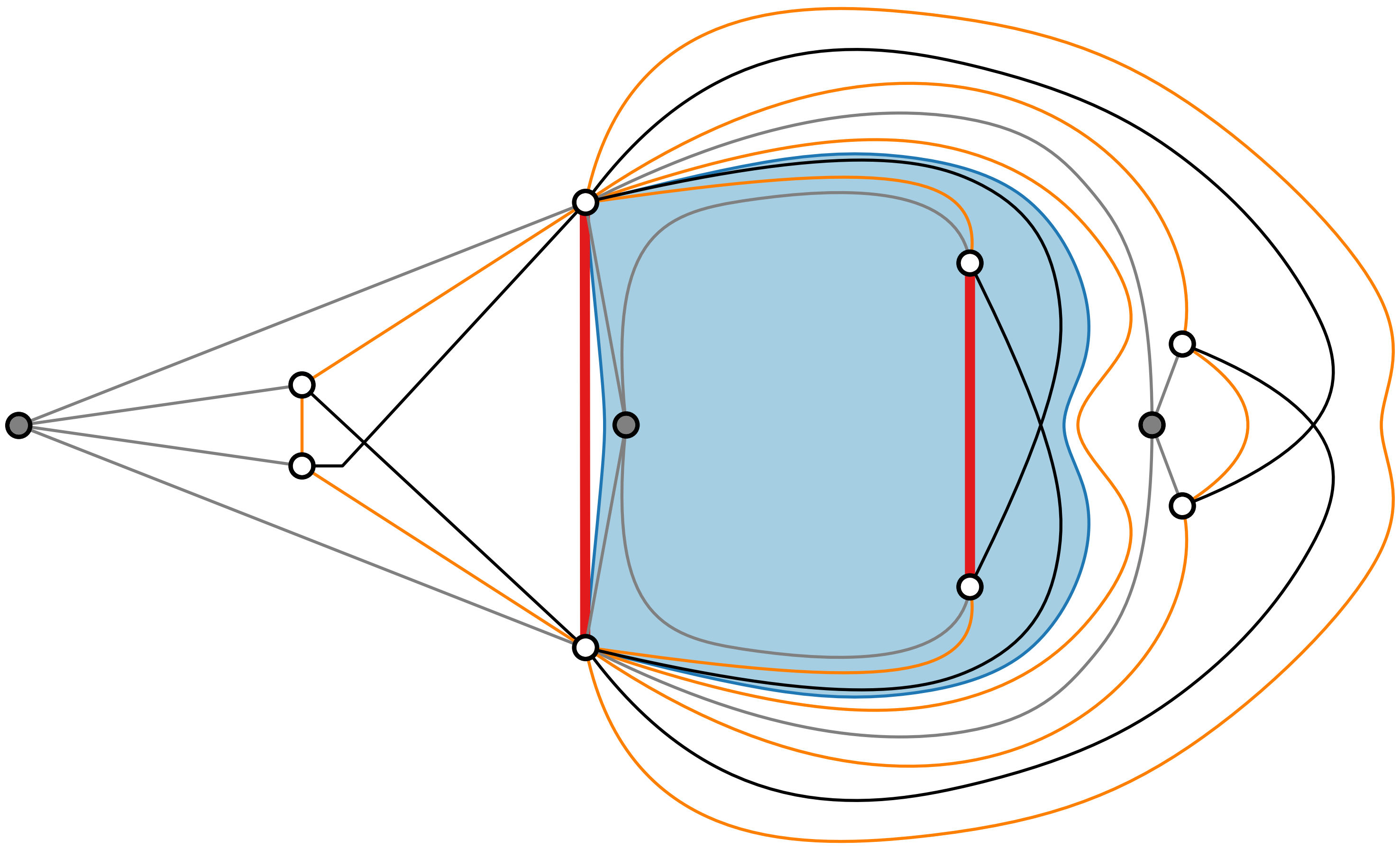
Algorithm Step 3: Drawing Procedure

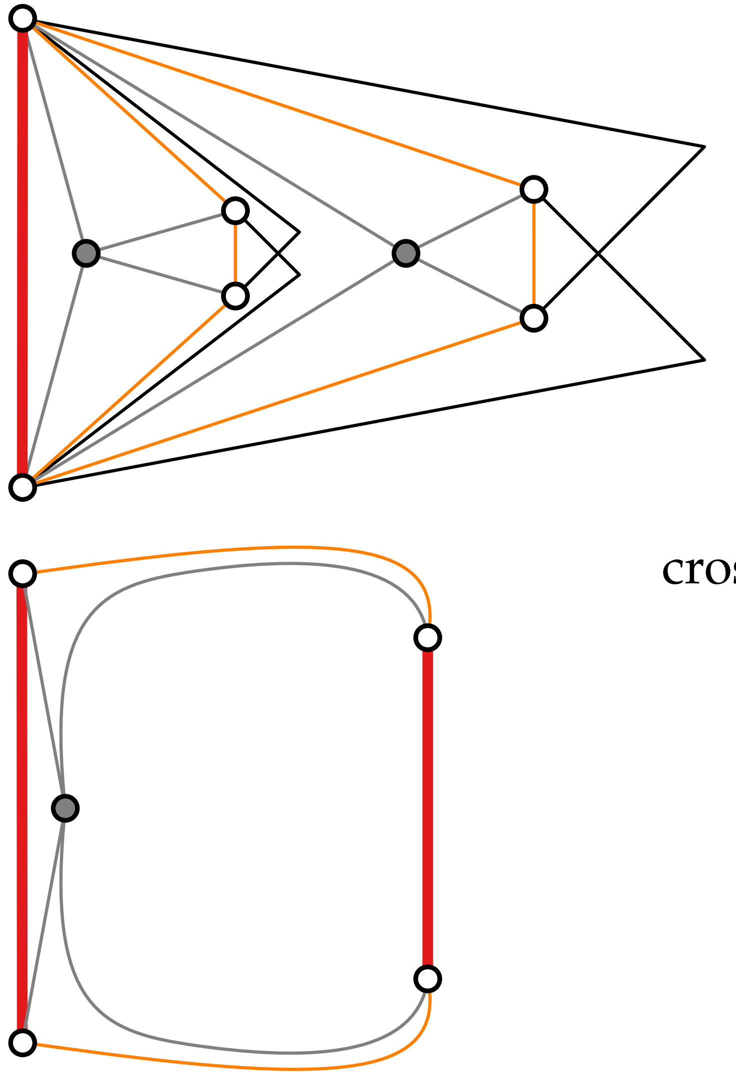




apply Chiba et al.

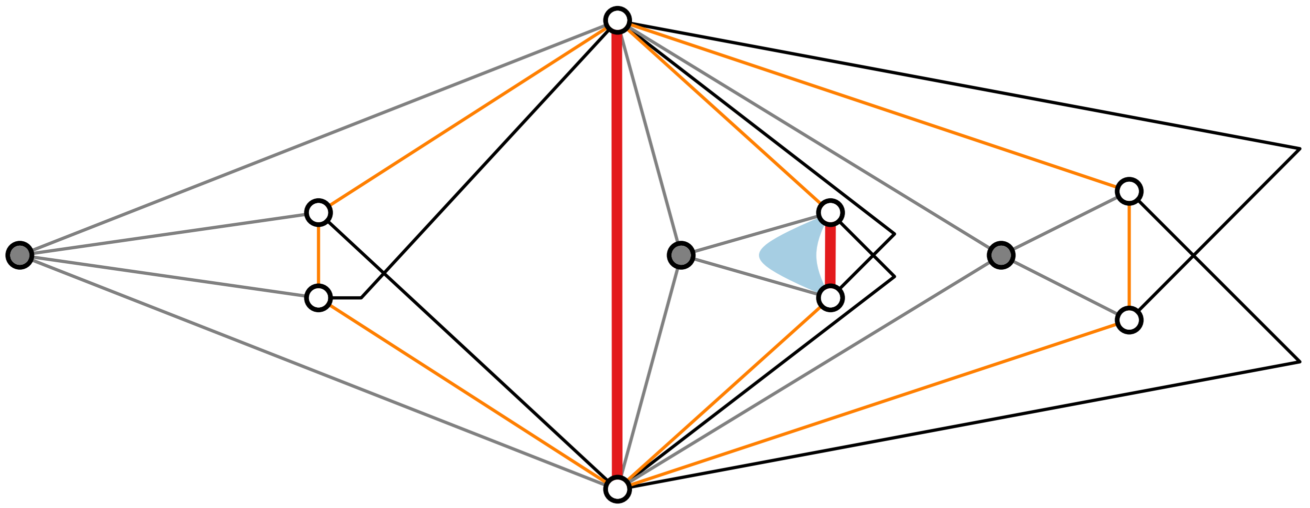
Algorithm Step 3: Drawing Procedure

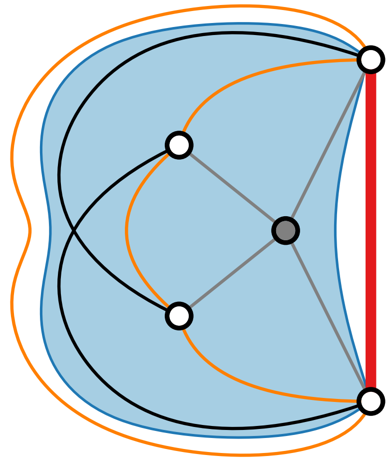




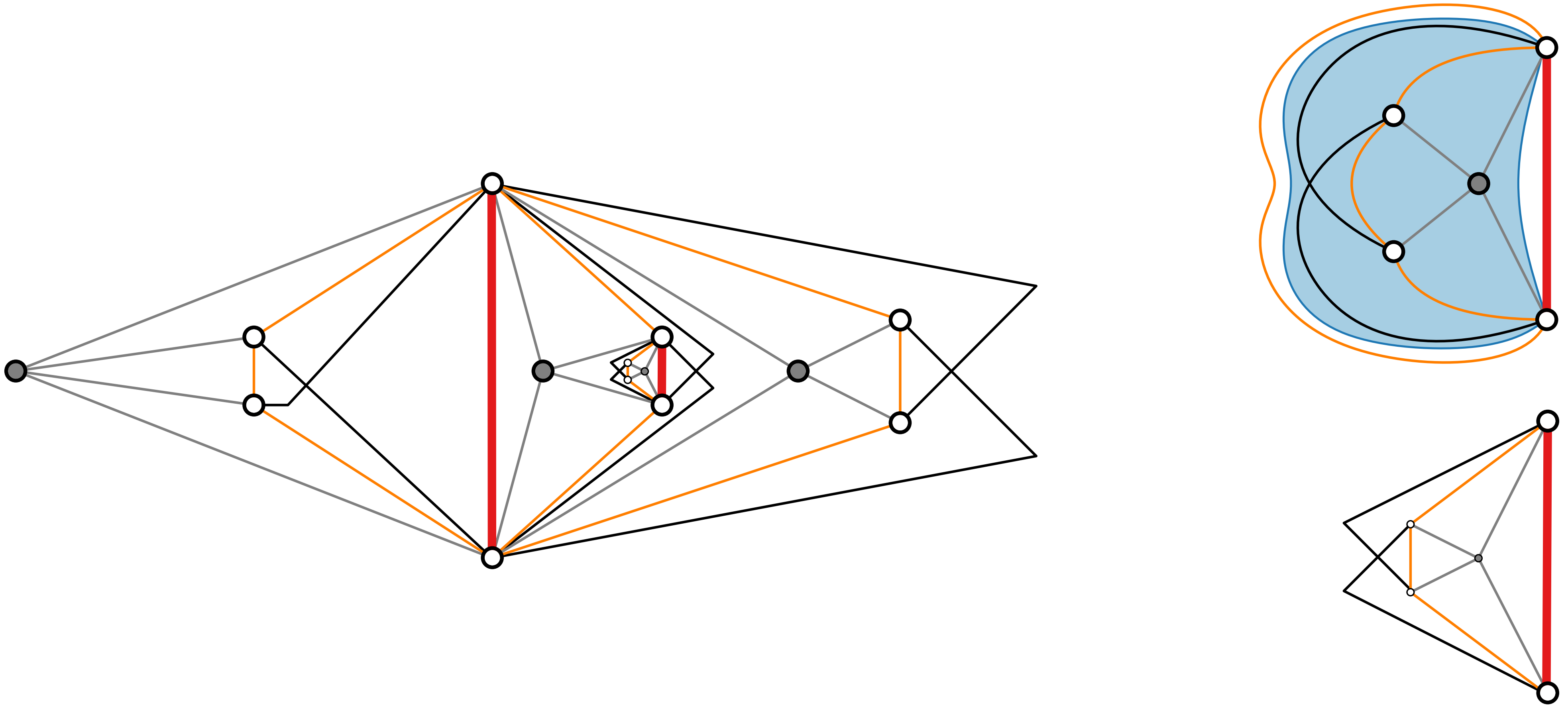
reinsert
crossing edges

Algorithm Step 3: Drawing Procedure



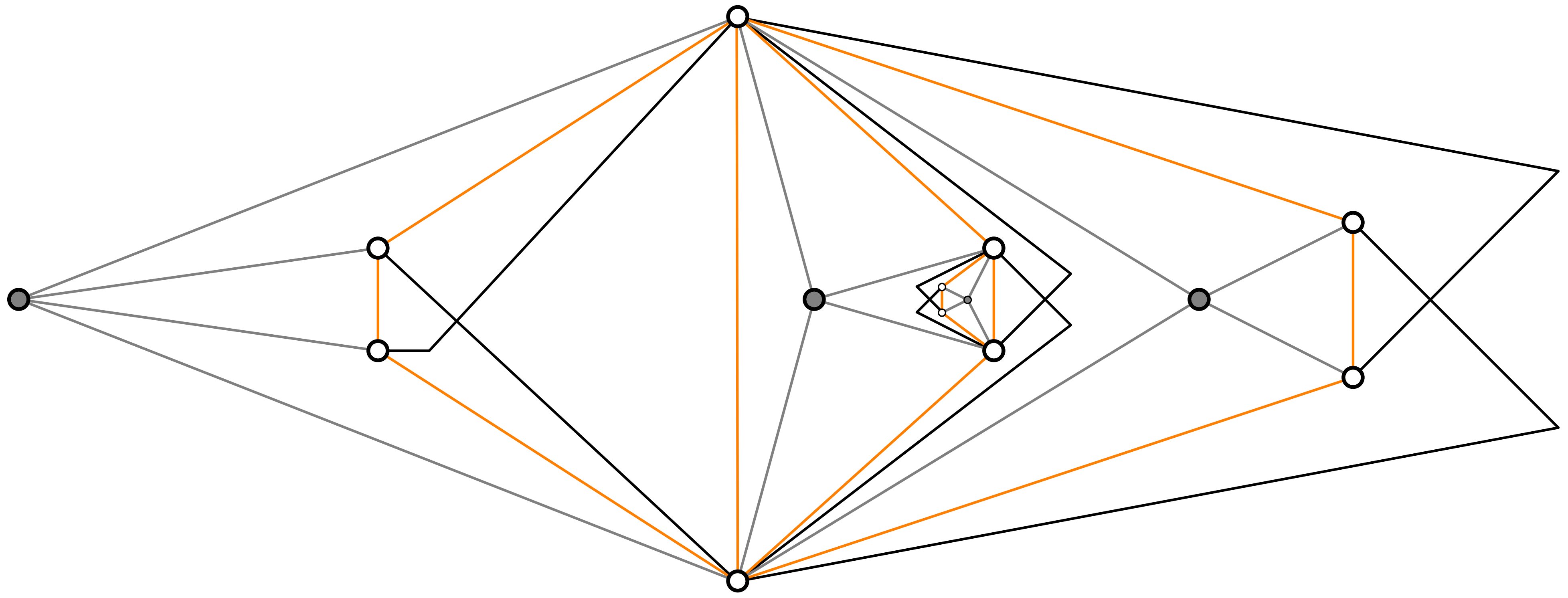


Algorithm Step 3: Drawing Procedure

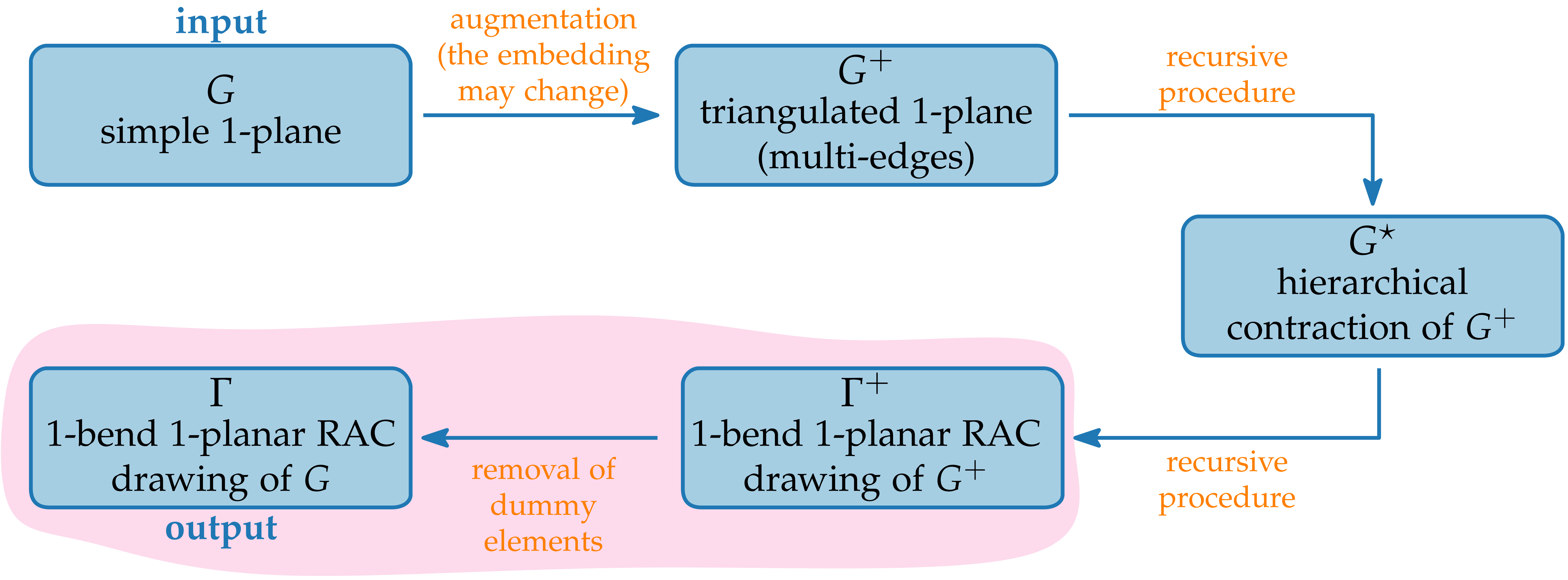


Algorithm Step 3: Drawing Procedure

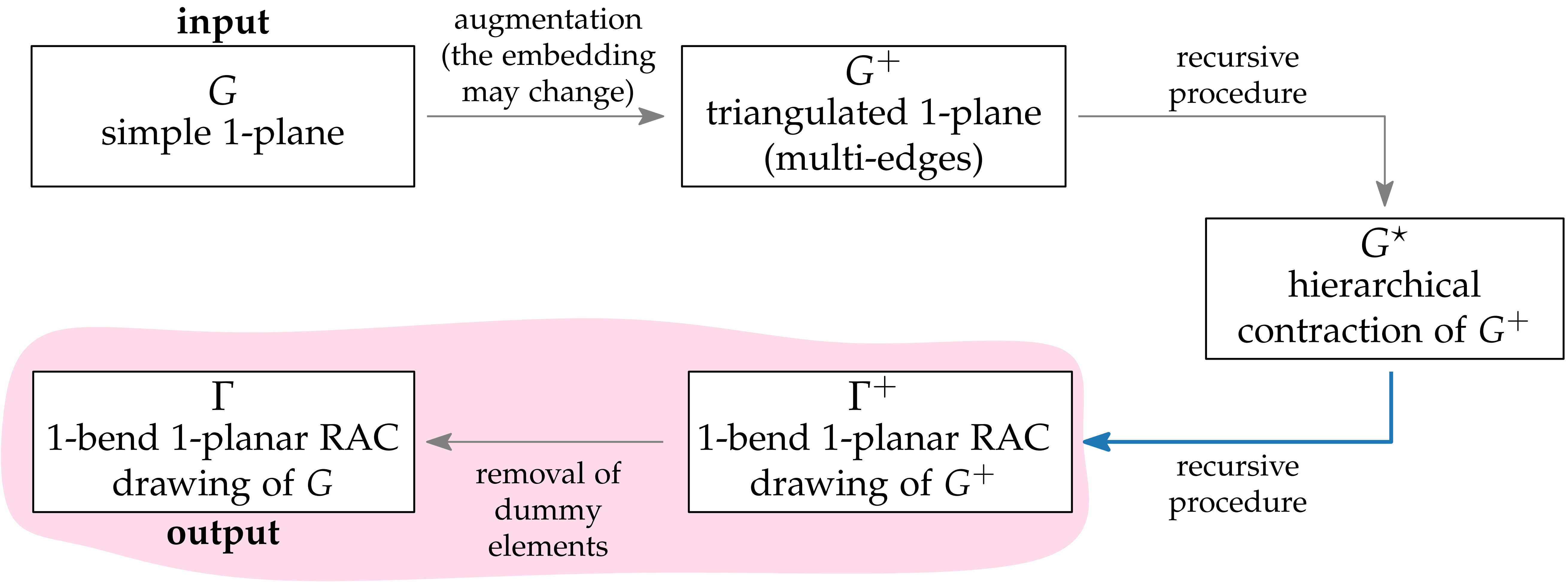
Γ^+
1-bend 1-planar RAC
drawing of G^+



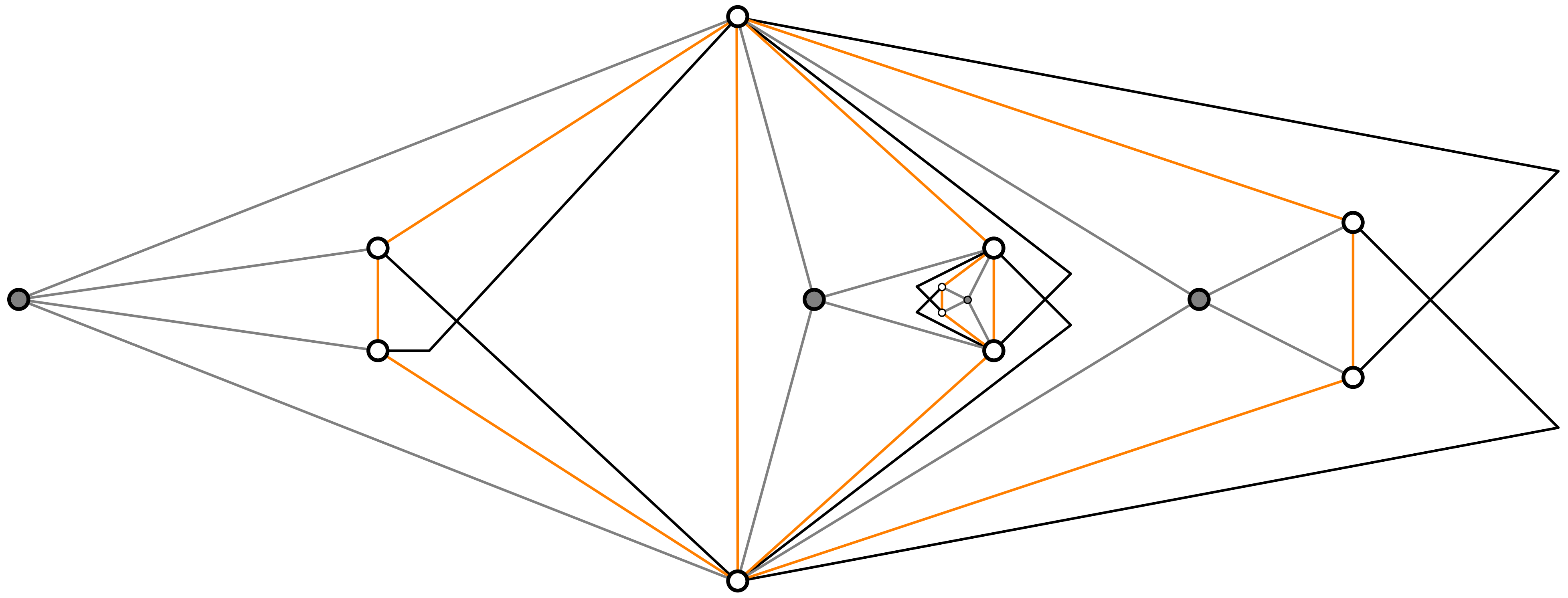
Algorithm Outline



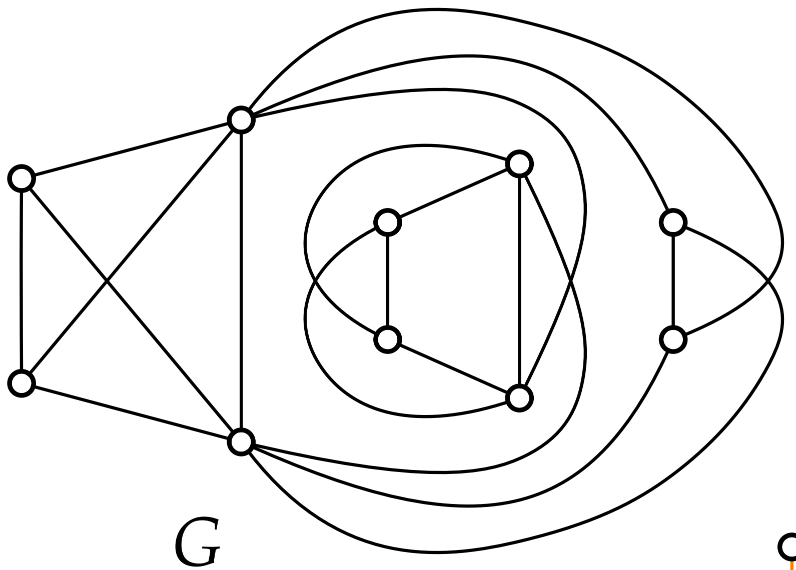
Algorithm Outline



Algorithm Step 4: Removal of Dummy Elements

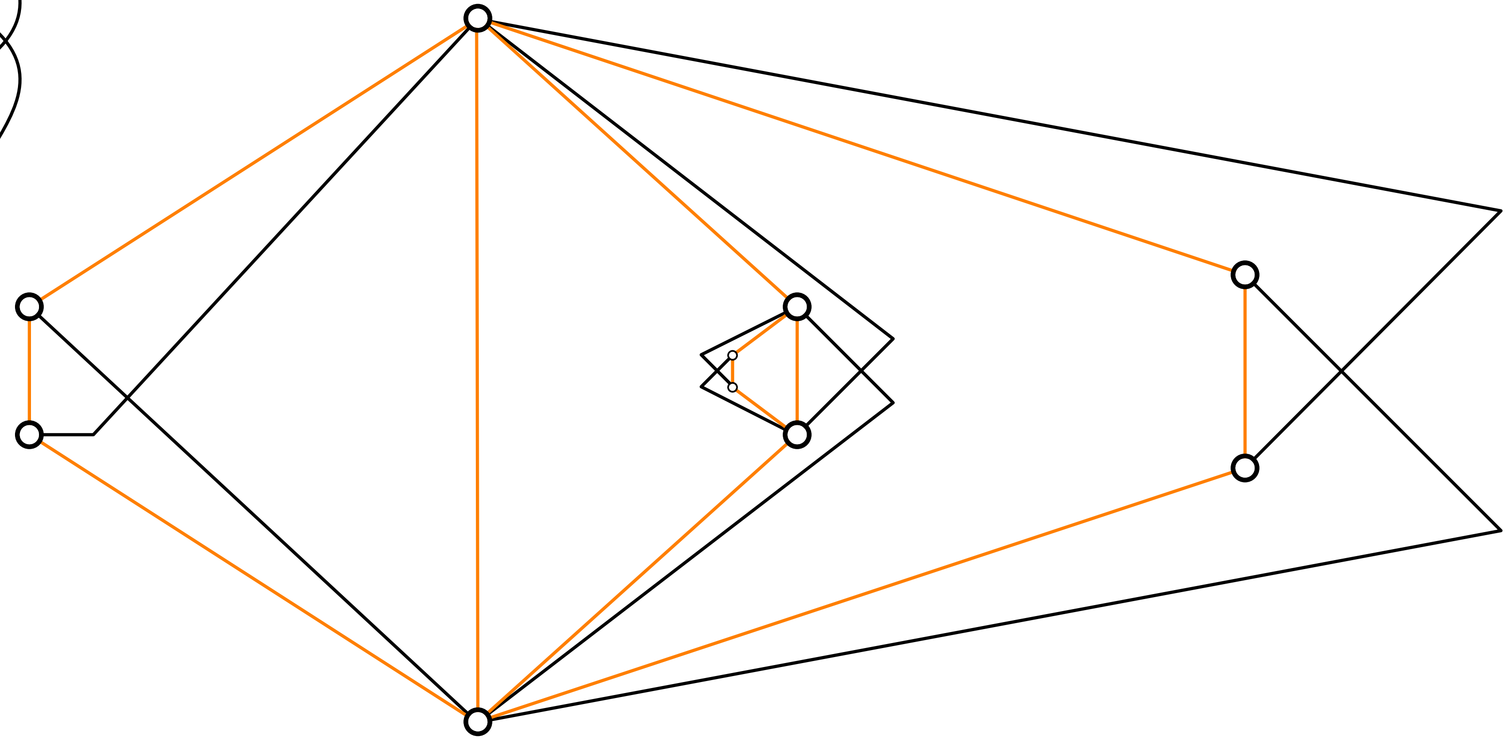


Algorithm Step 4: Removal of Dummy Elements

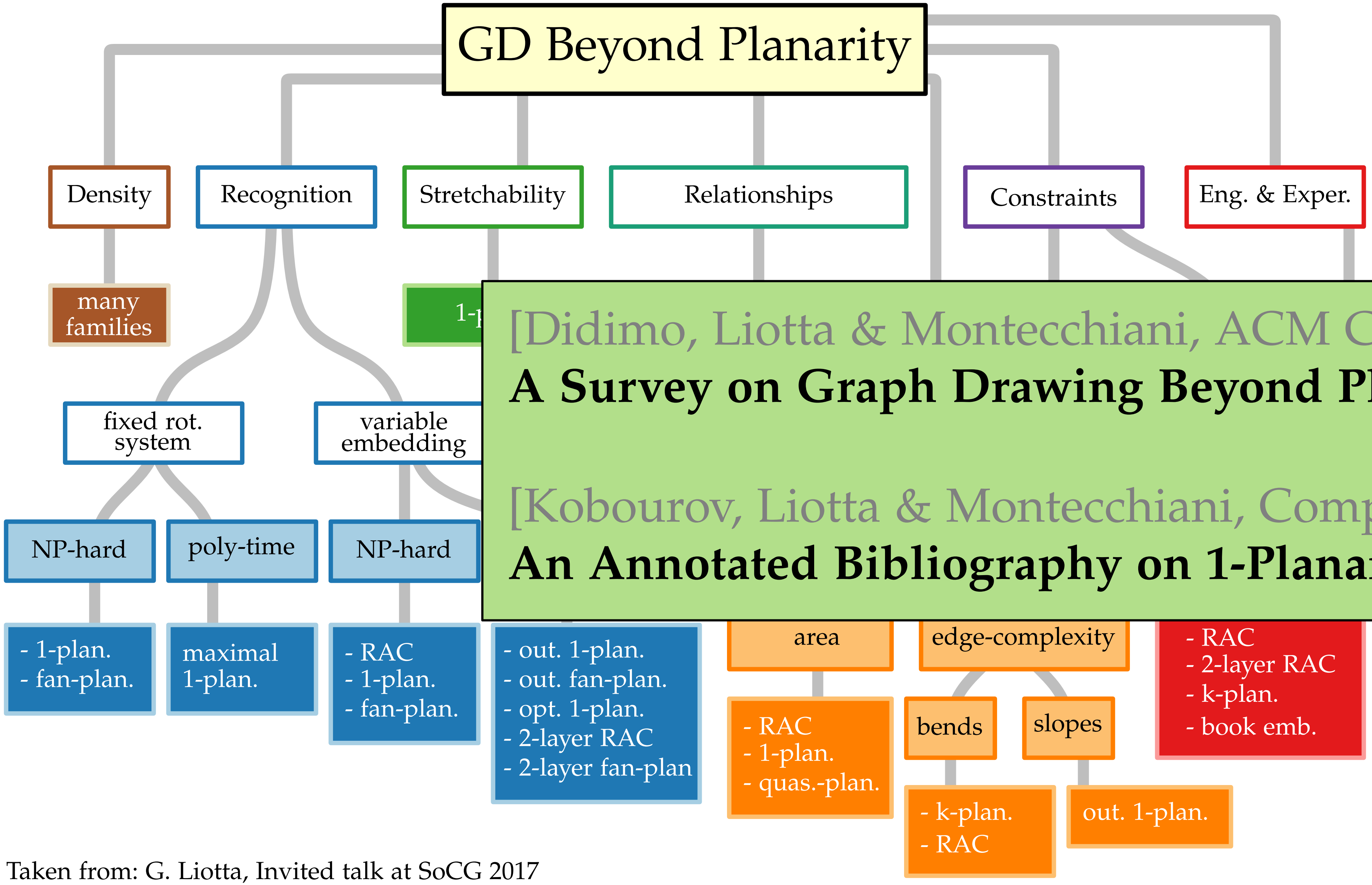


G
simple 1-plane

Γ
1-bend 1-planar RAC
drawing of G



GD Beyond Planarity: a Taxonomy



[Didimo, Liotta & Montecchiani, ACM Comput. Surv. 2019]
A Survey on Graph Drawing Beyond Planarity.

[Kobourov, Liotta & Montecchiani, Compu. Sci. Review 2017]
An Annotated Bibliography on 1-Planarity.

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017