

Visualization of Graphs

Lecture 5: Straight-Line Drawings of Planar Graphs II: Schnyder Realizer

Michael A. Bekos

Part I:
Barycentric Representation

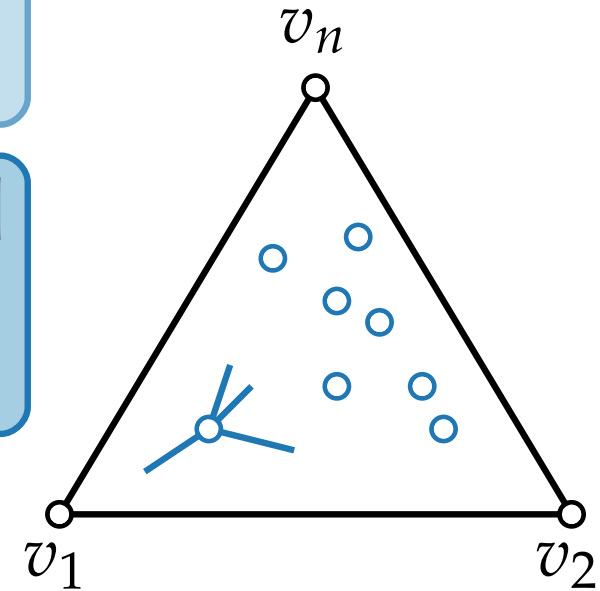
Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
 Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$.

Theorem. [Schnyder '89]
 Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 5) \times (2n - 5)$.

Idea.

- Fix outer triangle.
- Compute coordinates of inner vertices
 - based on outer triangle and
 - how much space there should be for other vertices
 - using weighted barycentric coordinates.

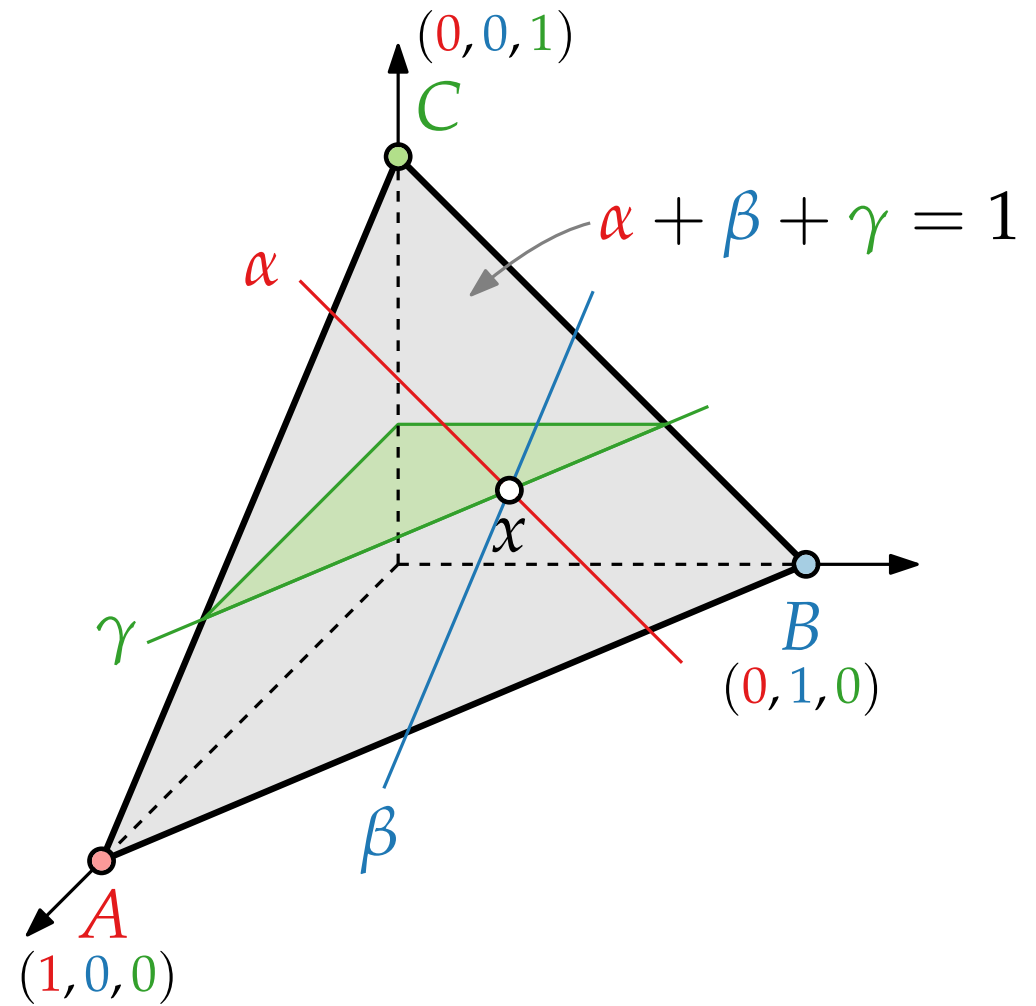
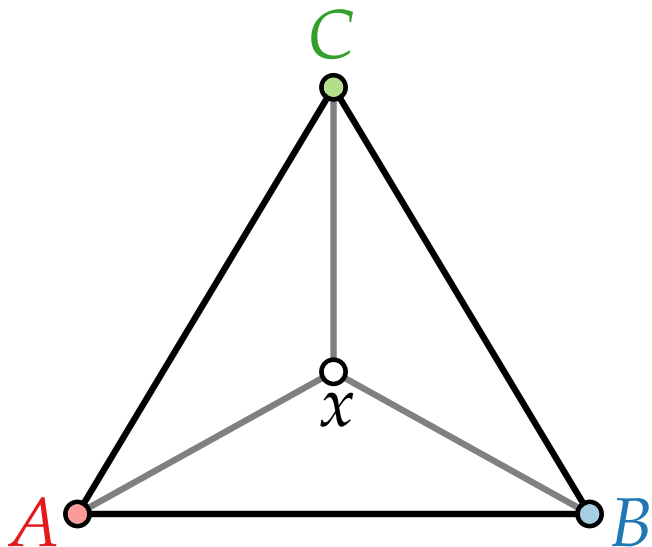


Barycentric Coordinates

Recall: $\text{barycenter}(x_1, \dots, x_k) = \sum_{i=1}^k x_i / k$

Let A, B, C form a triangle, let x lie inside $\triangle ABC$. The **barycentric coordinates** of x with respect to $\triangle ABC$ are a triple $(\alpha, \beta, \gamma) \in \mathbb{R}_{\geq 0}^3$ such that

- $x = \alpha A + \beta B + \gamma C$ and
- $\alpha + \beta + \gamma = 1$

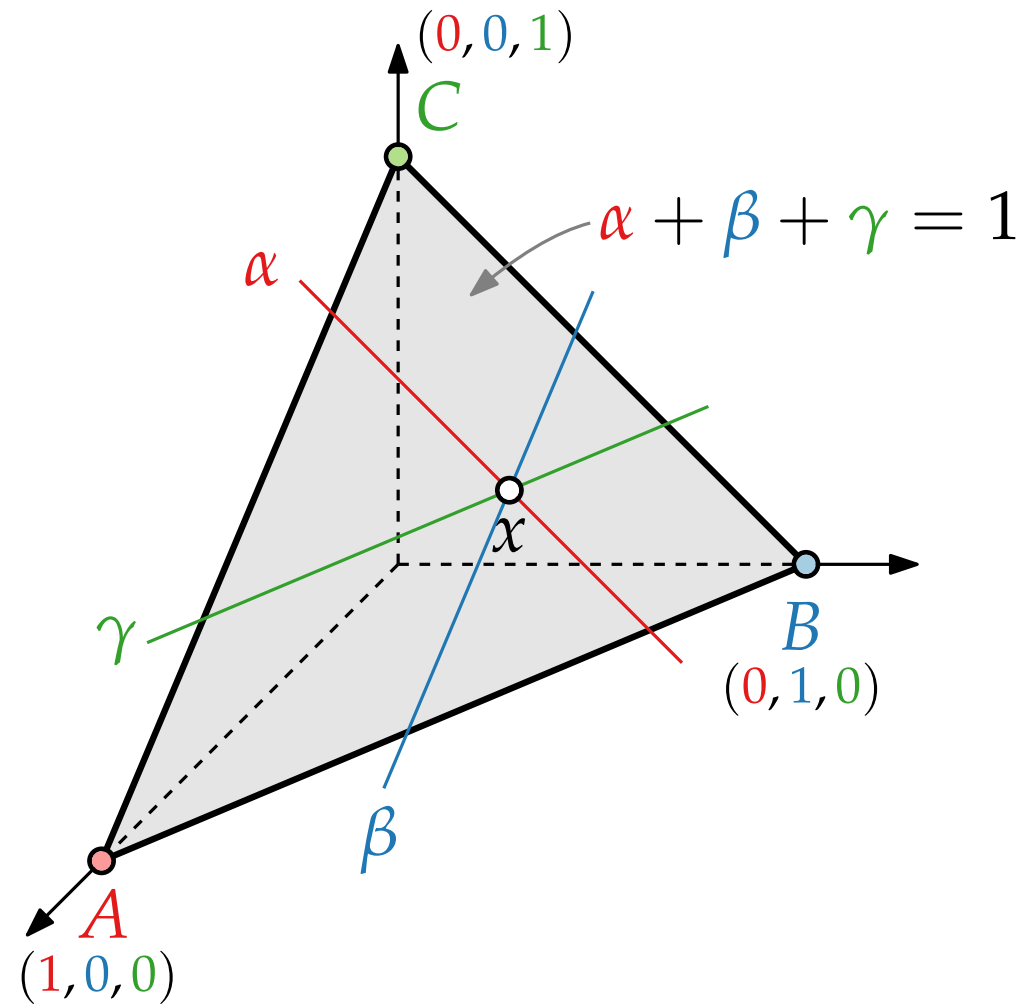
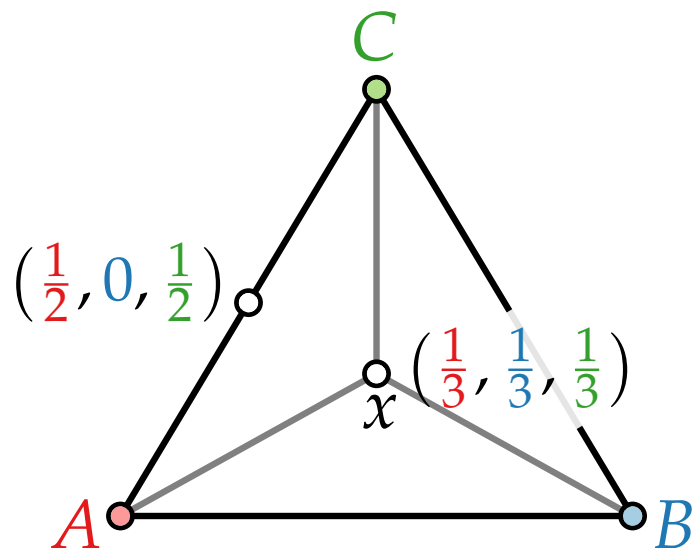


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Barycentric Representation

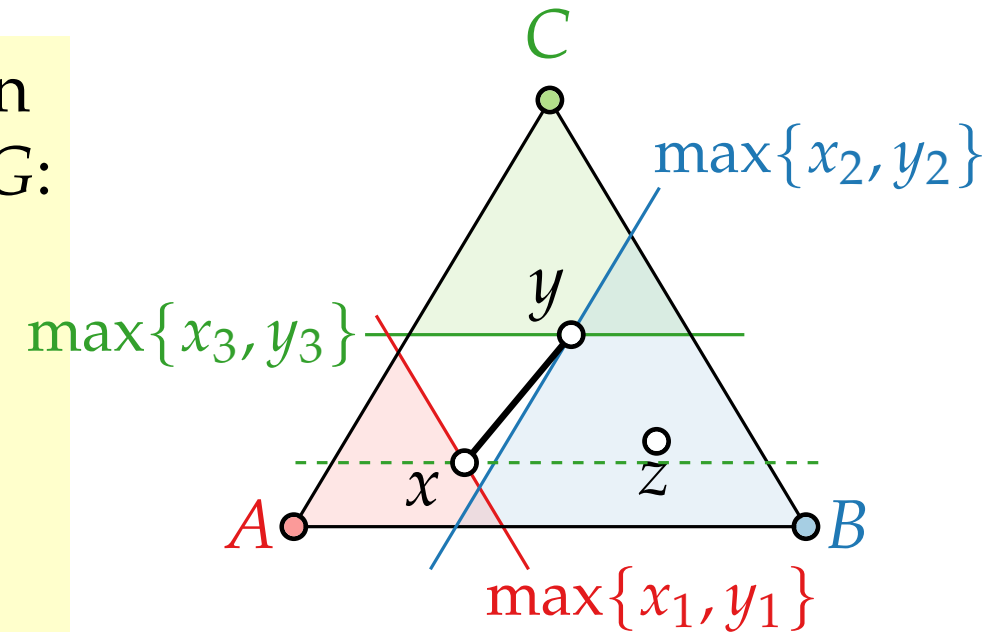
A **barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of G :

$$f: V \rightarrow \mathbb{R}_{\geq 0}^3, v \mapsto (v_1, v_2, v_3)$$

with the following properties:

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V$,

(B2) for each $xy \in E$ and each $z \in V \setminus \{x, y\}$
there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$.



Barycentric Representation

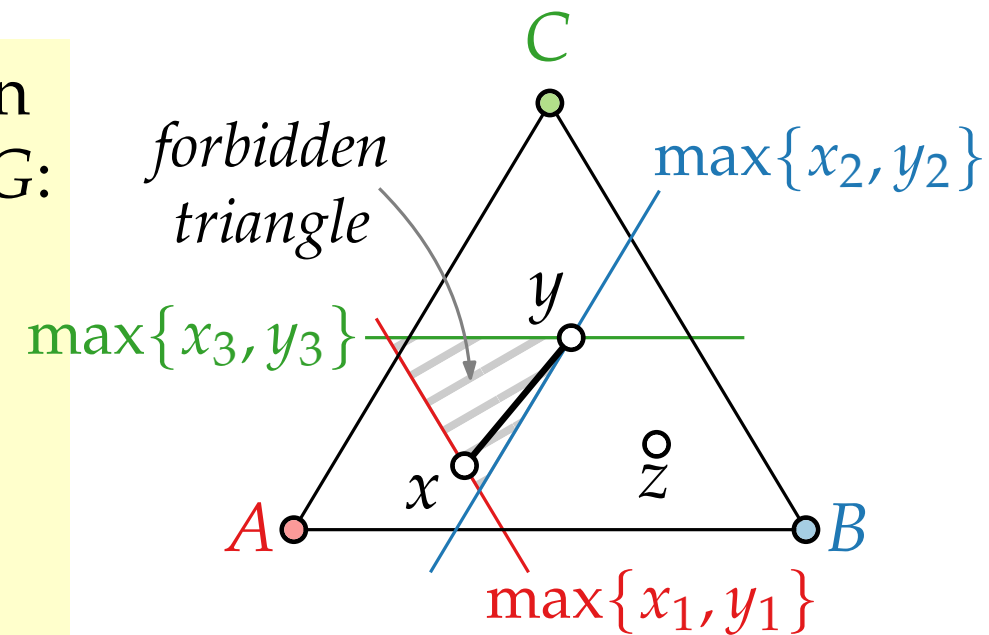
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Barycentric Representations of Planar Graphs

How to find barycentric representation?

Lemma.

Let $f : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G and let $A, B, C \in \mathbb{R}^2$ be in general position. Then the mapping

$$\phi : v \in V \mapsto v_1 A + v_2 B + v_3 C$$

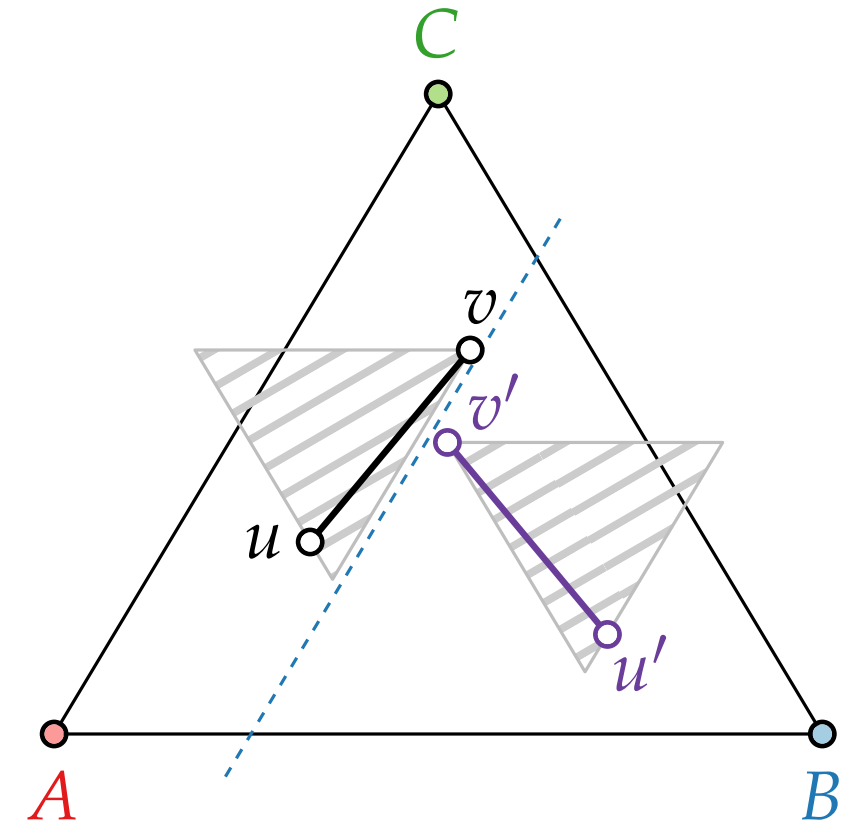
gives a **planar** drawing of G inside $\triangle ABC$.

- No vertex x can lie on an edge $\{u, v\}$.
- No pair of edges $\{u, v\}$ and $\{u', v'\}$ cross:

$$u'_i > u_i, v_i \quad v'_j > u_j, v_j \quad u_k > u'_k, v'_k \quad v_l > u'_l, v'_l$$

$$\Rightarrow \{i, j\} \cap \{k, l\} = \emptyset$$

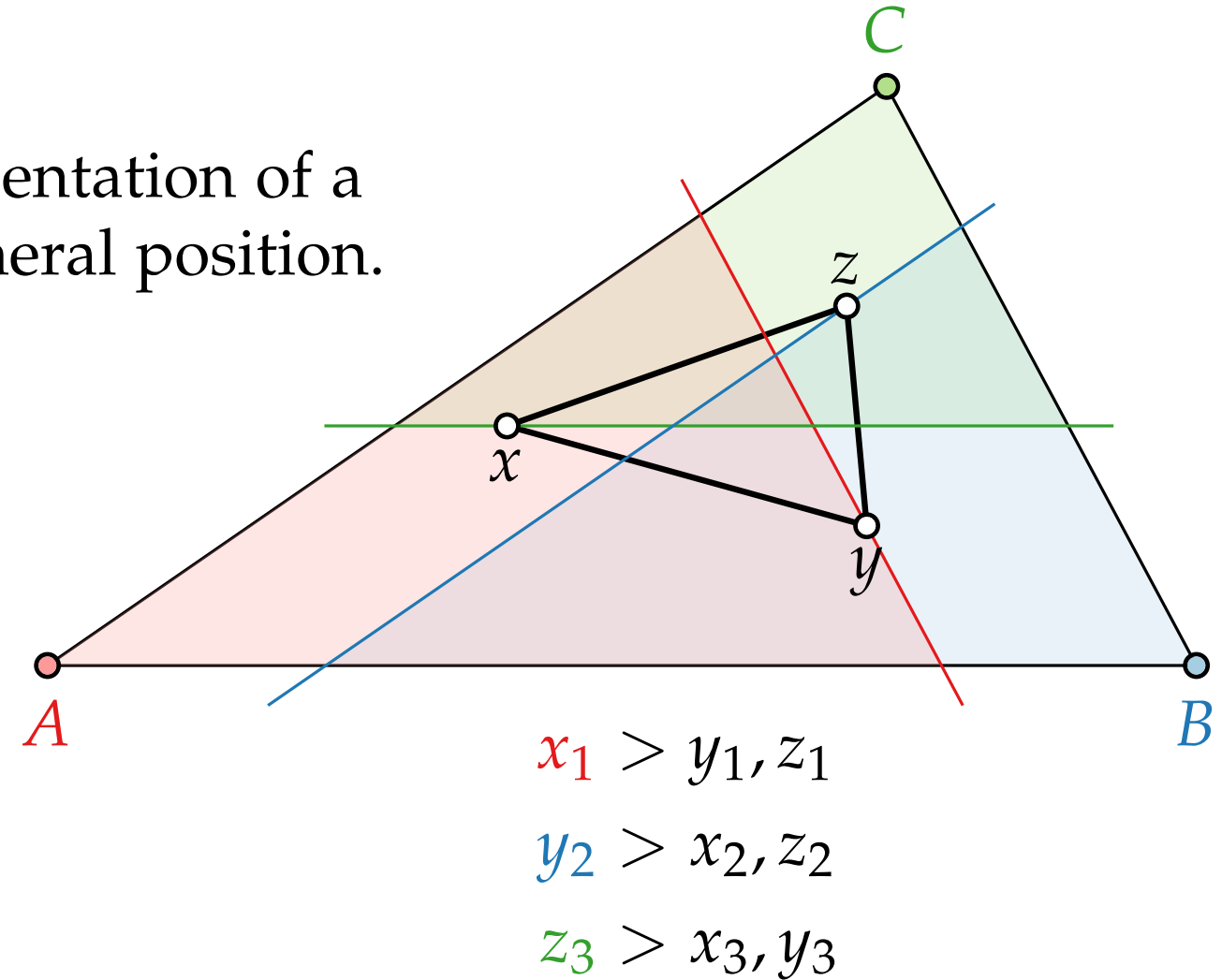
$$\text{wlog } i = j = 2 \Rightarrow u'_2, v'_2 > u_2, v_2 \Rightarrow \text{separated by straight line}$$



Part II: Schnyder Realizer

Schnyder Labeling

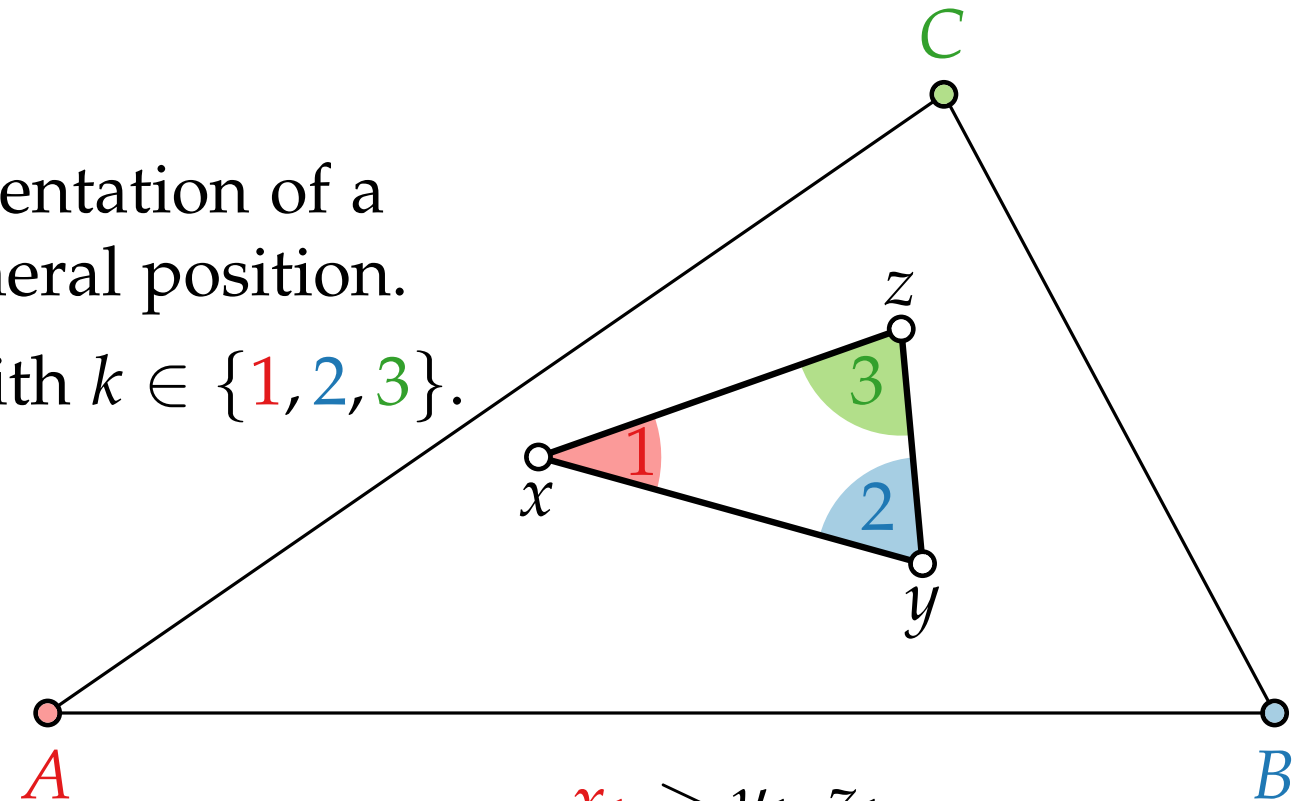
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Schnyder Labeling

Let $\phi : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G and let $A, B, C \in \mathbb{R}^2$ be in general position.

We can label each angle in $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.



$$x_1 > y_1, z_1$$

$$y_2 > x_2, z_2$$

$$z_3 > x_3, y_3$$

Schnyder Labeling

Let $\phi : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G and let $A, B, C \in \mathbb{R}^2$ be in general position.

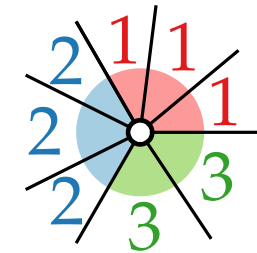
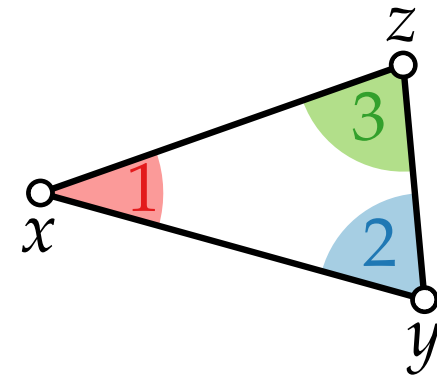
We can label each angle in $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.

A **Schnyder Labeling** of a plane triangulation G is a labeling of all internal angles with labels **1**, **2** and **3** such that:

Faces: The three angles of an internal face are labeled **1**, **2** and **3** in counterclockwise order.

Vertices: The ccw order of labels around each vertex consists of

- a nonempty interval of **1**'s
- followed by a nonempty interval of **2**'s
- followed by a nonempty interval of **3**'s.

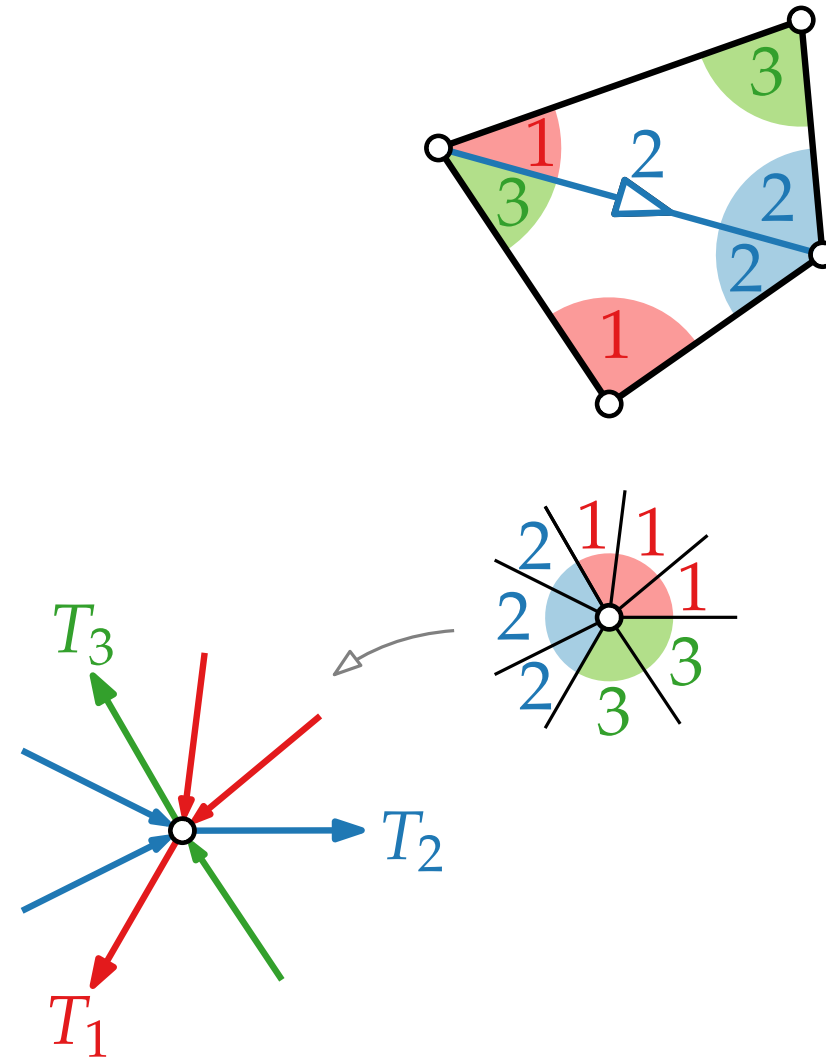


Schnyder Realizer

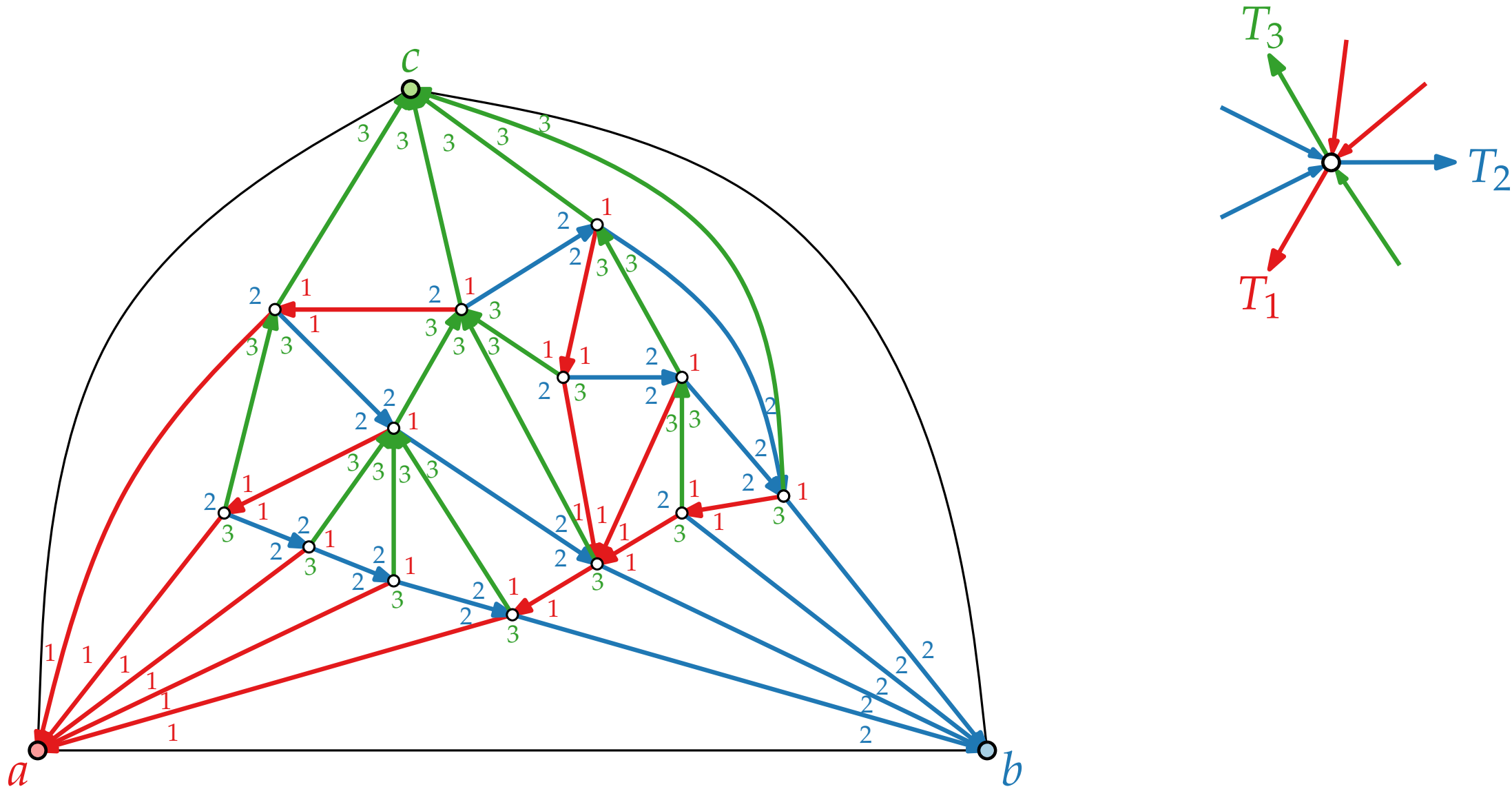
A Schnyder labeling induces an edge labeling.

A **Schnyder Realizer** (or **Wood**) of a plane triangulation $G = (V, E)$ is a partition of the inner edges of E into three sets of oriented edges T_1, T_2, T_3 such that for each inner vertex $v \in V$ holds:

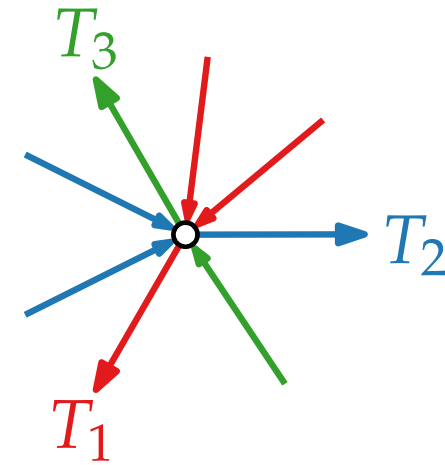
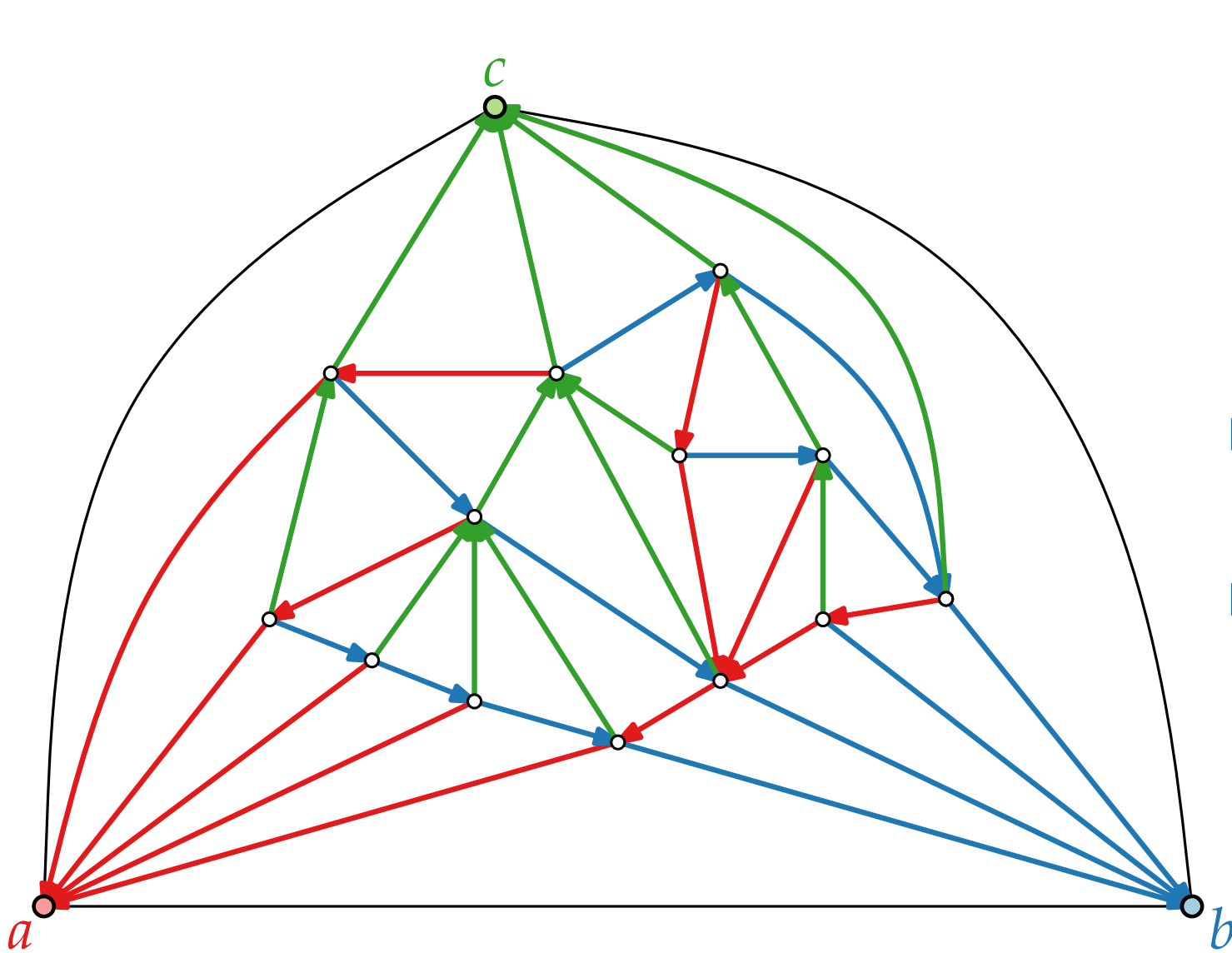
- v has one outgoing edge in each of $T_1, T_2,$ and T_3 .
- The ccw order of edges around v is:
 leaving in T_1 , entering in T_3 , leaving in T_2 ,
 entering in T_1 , leaving in T_3 , entering in T_2 .



Schnyder Realizer – Example and Properties



Schnyder Realizer – Example and Properties



- All inner edges incident to a , b , and c are incoming in the same color.
- T_1 , T_2 , and T_3 are trees on all inner vertices and one outer vertex each (as its root).

Schnyder Realizer – Existence

Lemma.

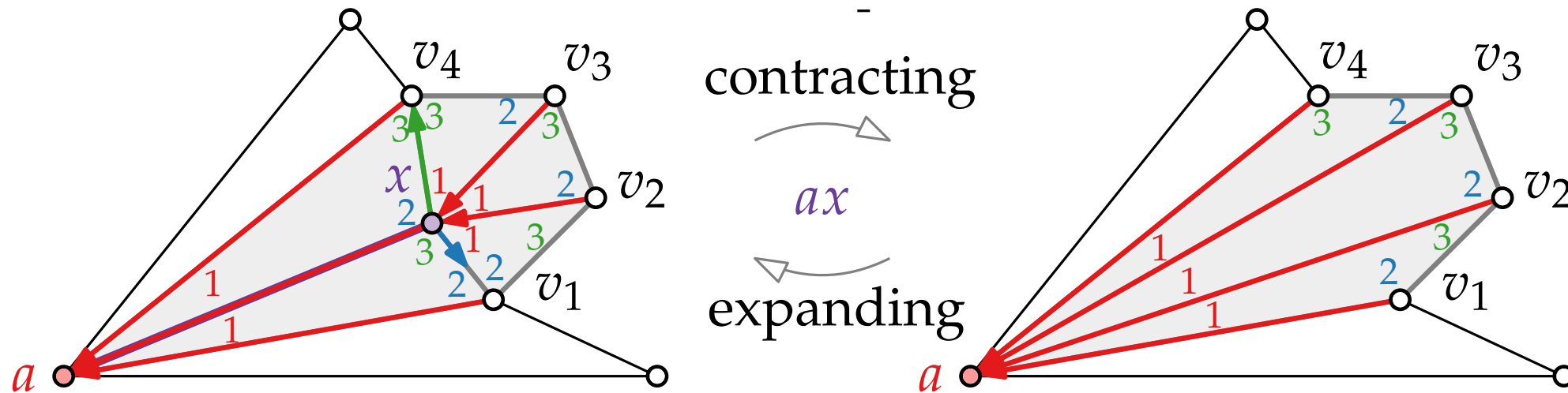
[Kampen 1976]

Let G be a plane triangulation with vertices a, b, c on the outer face. There exists a **contractible edge** $\{a, x\}$ in G , $x \neq b, c$.

Theorem.

Every plane triangulation has a Schnyder Labeling and Realizer.

Proof by induction on # vertices via edge contractions.

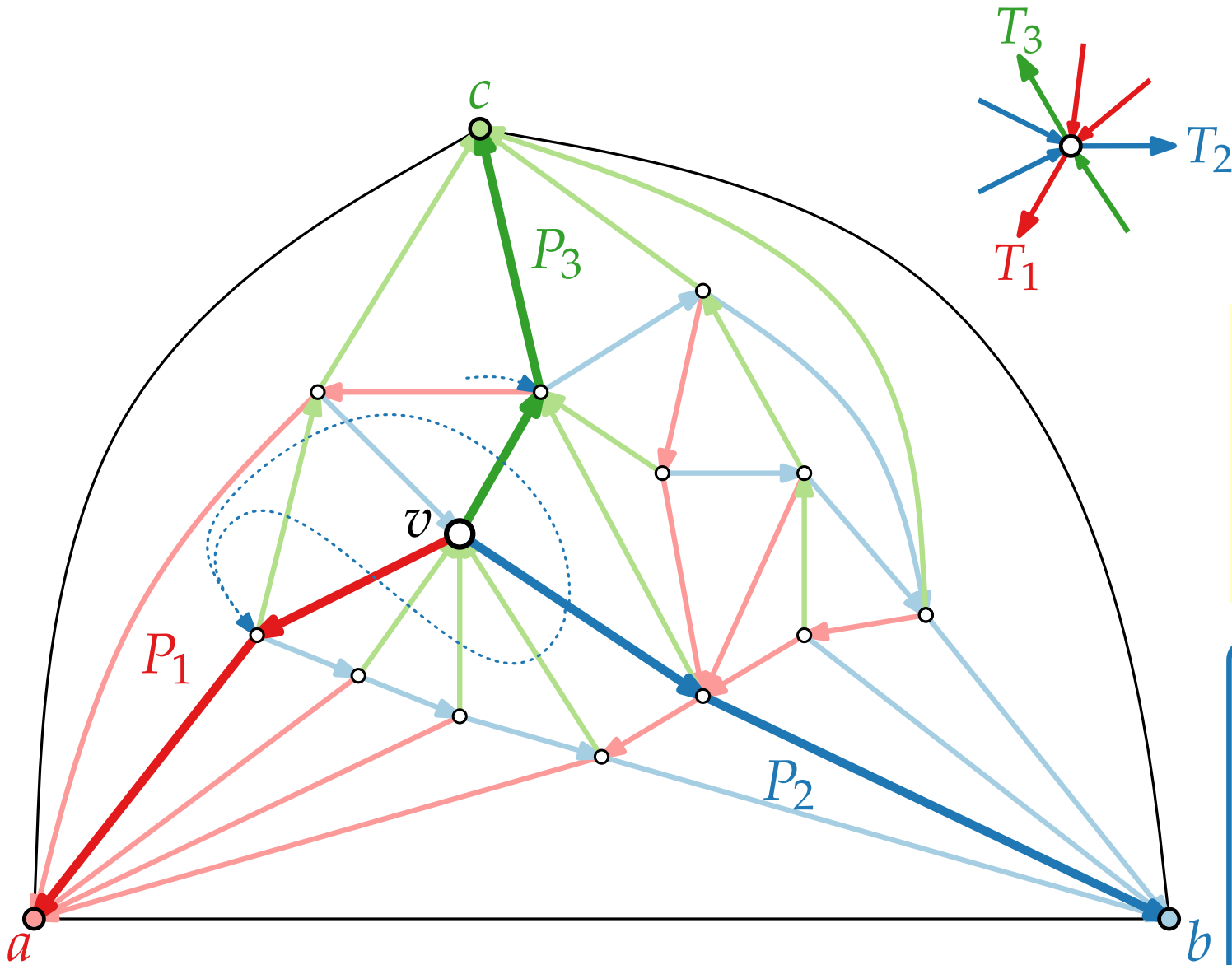


Constructive proof can be used as algorithm to compute a Schnyder labeling. It can be implemented in $\mathcal{O}(n)$ time

... requires that a and x have exactly 2 common neighbors.

Part III:
Schnyder Drawings

Schnyder Realizer – More Properties



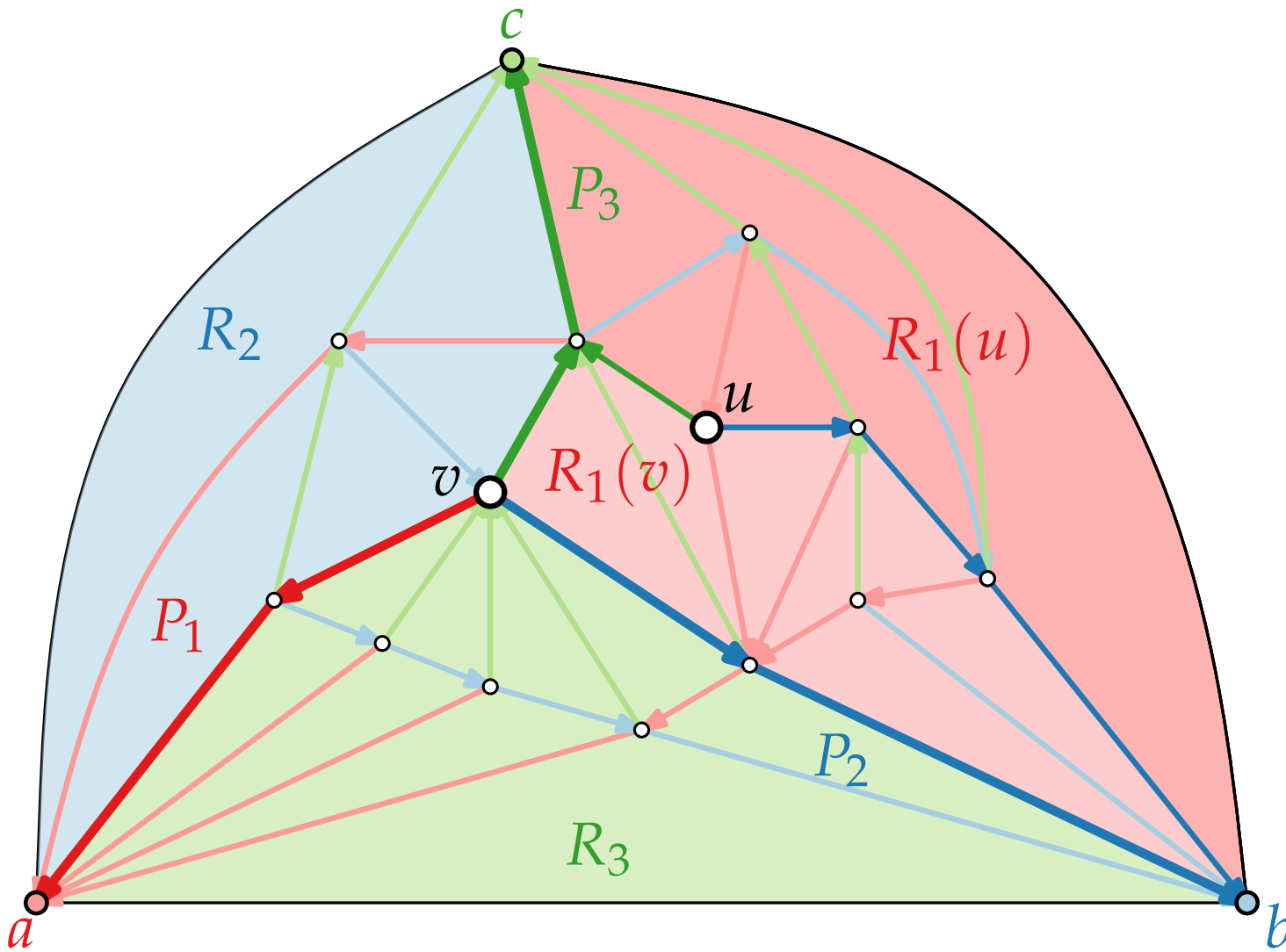
- From each vertex v there exists a directed **red** path $P_1(v)$ to a , a directed **blue** path $P_2(v)$ to b , and a directed **green** path $P_3(v)$ to c .

$P_i(v)$: path from v to root of T_i .

Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v .

Schnyder Realizer – More Properties



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$P_i(v)$: path from v to root of T_i .

$R_1(v)$: set of faces contained in P_2, bc, P_3 .

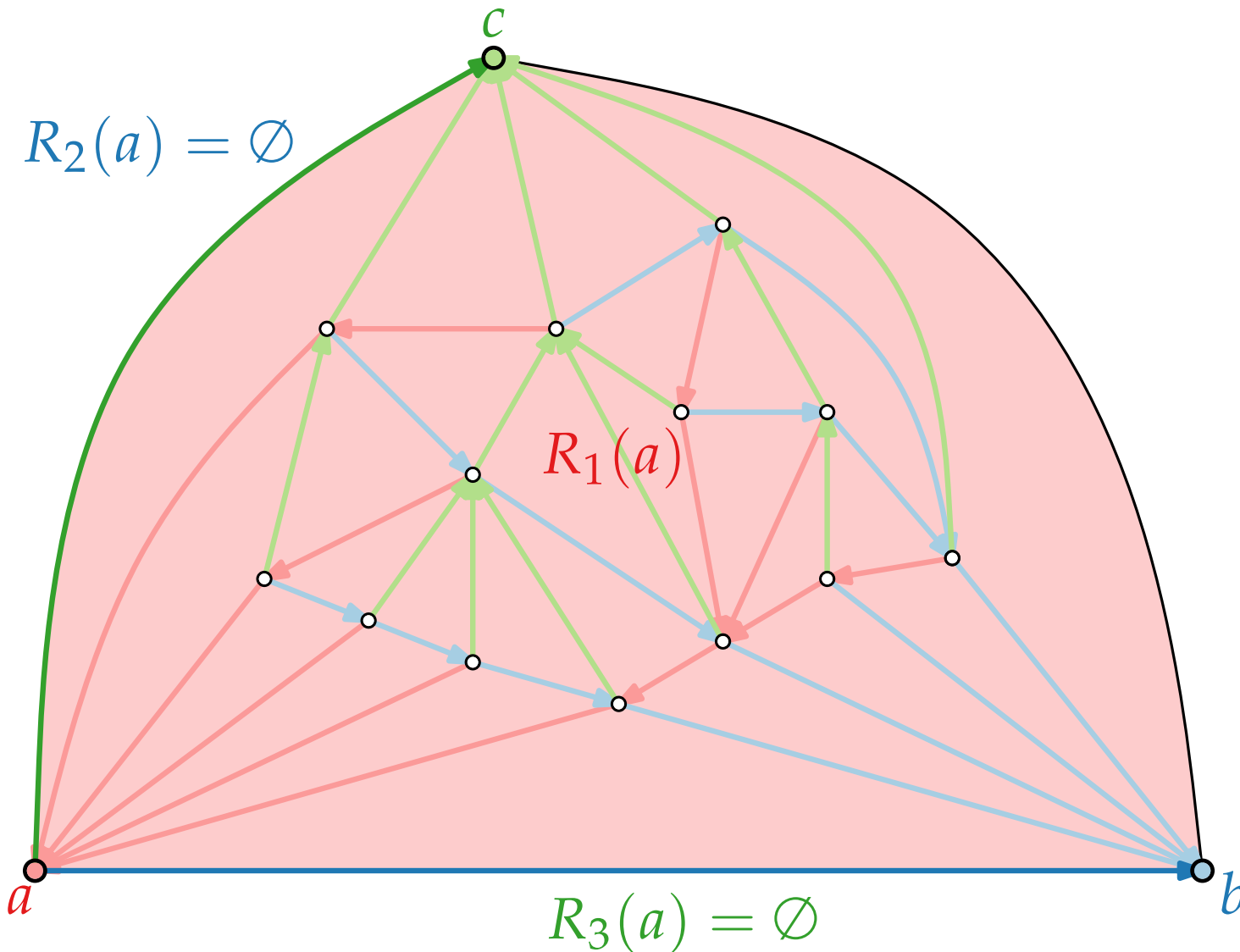
$R_2(v)$: set of faces contained in P_3, ca, P_1 .

$R_3(v)$: set of faces contained in P_1, ab, P_2 .

Lemma.

- $P_1(v), P_2(v), P_3(v)$ cross only at v .
- For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

Schnyder Realizer – More Properties



- From each vertex v there exists a directed **red** path $P_1(v)$ to a , a directed **blue** path $P_2(v)$ to b , and a directed **green** path $P_3(v)$ to c .

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Lemma.

- $P_1(v), P_2(v), P_3(v)$ cross only at v .
- For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.
- $|R_1(v)| + |R_2(v)| + |R_3(v)| = 2n - 5$

Schnyder Drawing

Set $A = (0, 0)$, $B = (2n - 5, 0)$, and $C = (0, 2n - 5)$.

Theorem.

[Schnyder '89]

For a plane triangulation G , the mapping

$$f: v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5} (|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

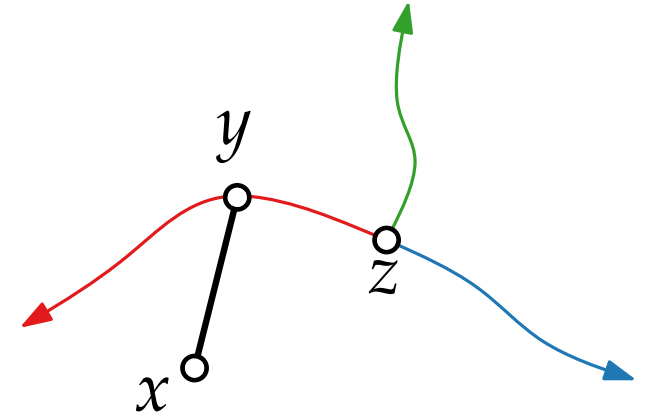
is a barycentric representation of G , which thus gives a planar straight-line drawing of G on the $(2n - 5) \times (2n - 5)$ grid.

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V$ ✓

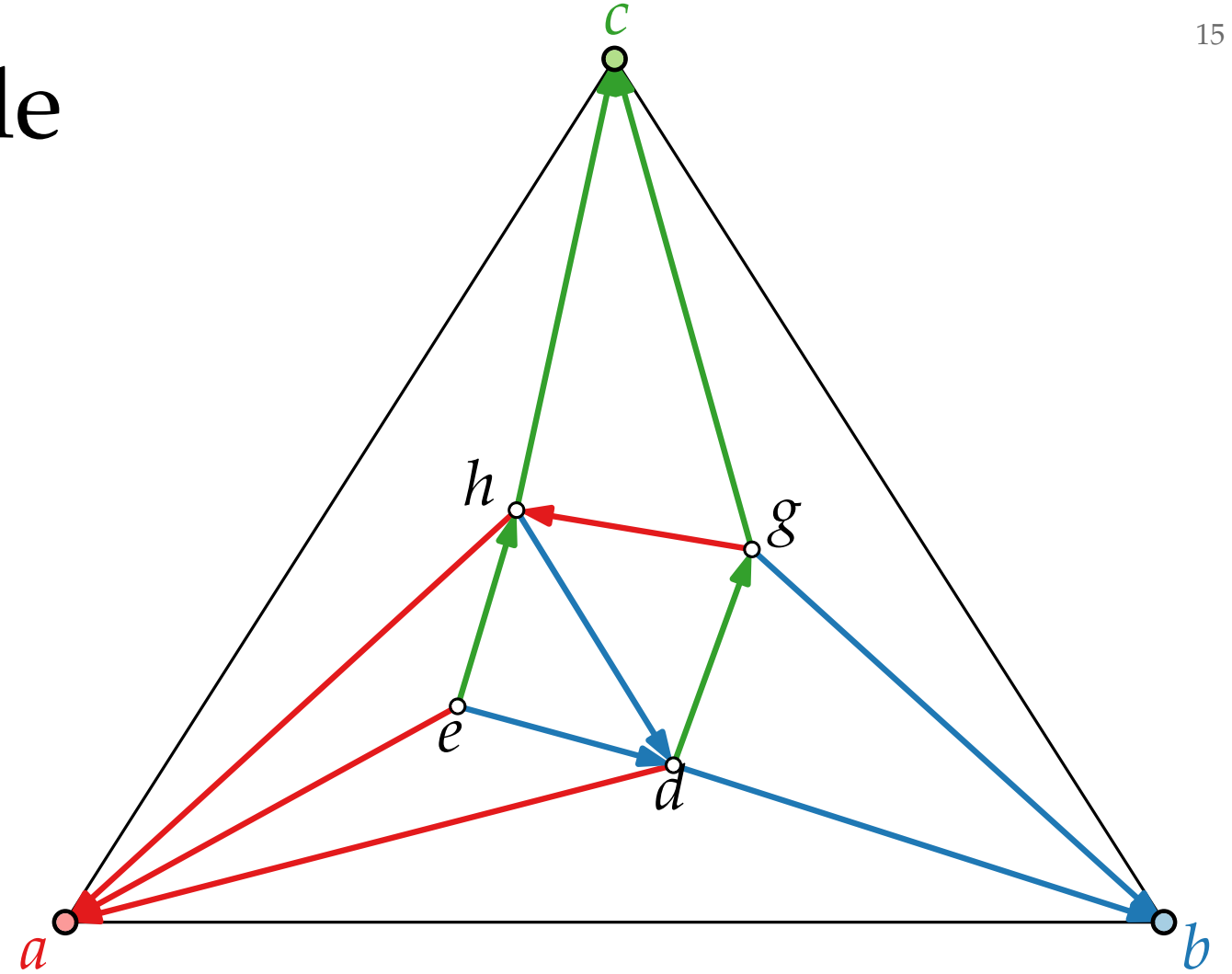
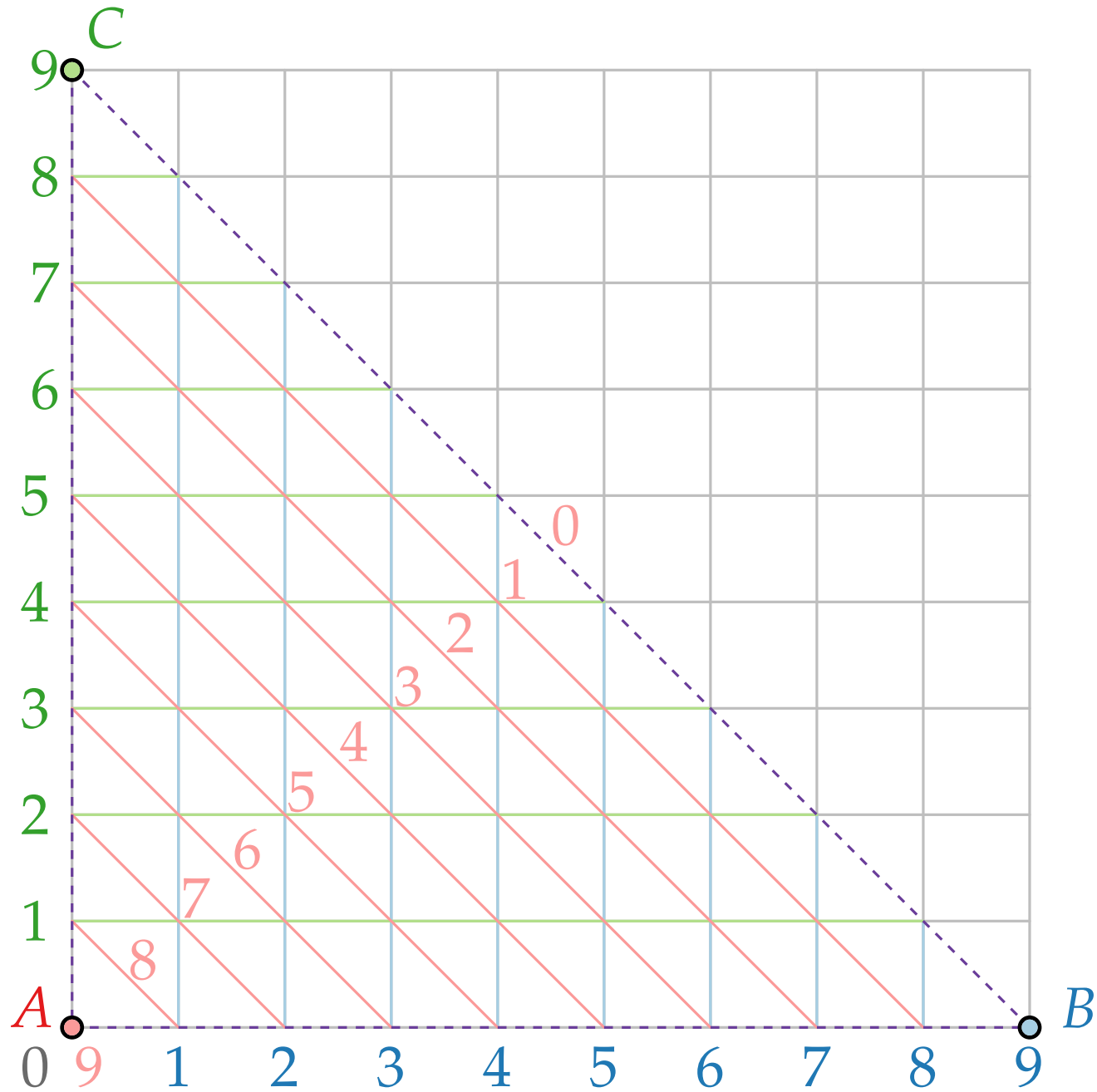
(B2) for each $\{x, y\} \in E$ and each $z \in V \setminus \{x, y\}$
there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$ ✓

- $\{x, y\}$ must lie in some $R_i(z)$ for $i \in \{1, 2, 3\}$

- For inner vertices $u \neq v$ it holds
that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.



Schnyder Drawing – Example



$$n = 7, 2n - 5 = 9 \quad f(d) = (2, 6, 1)$$

$$f(a) = (9, 0, 0) \quad f(e) = (5, 2, 2)$$

$$f(b) = (0, 9, 0) \quad f(g) = (1, 2, 6)$$

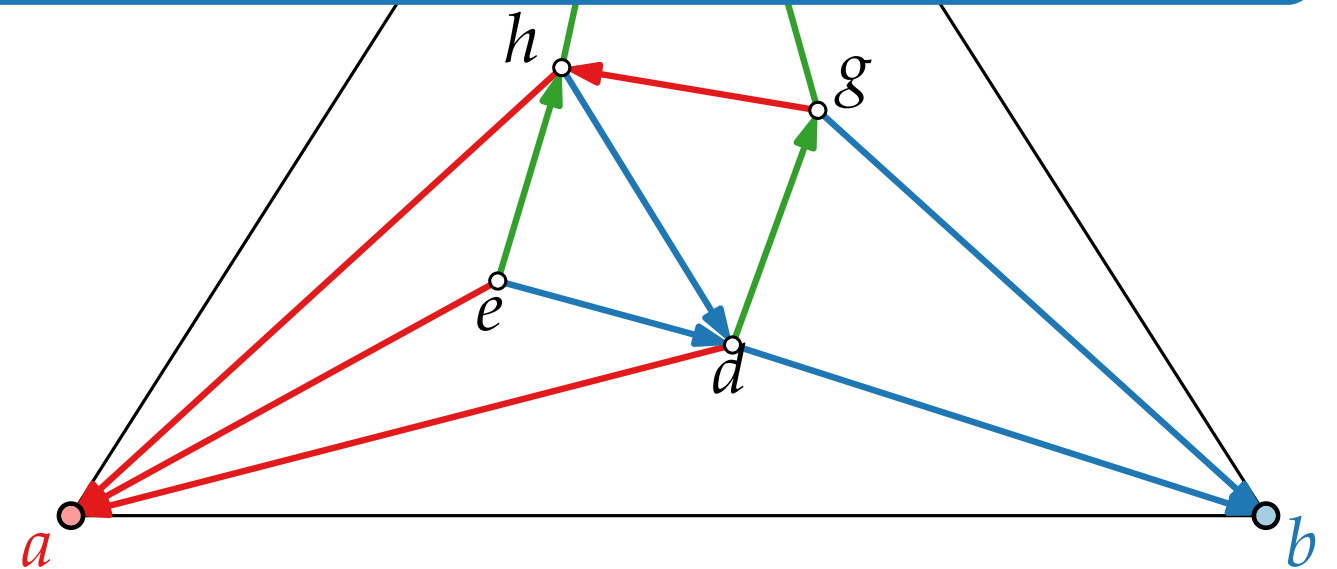
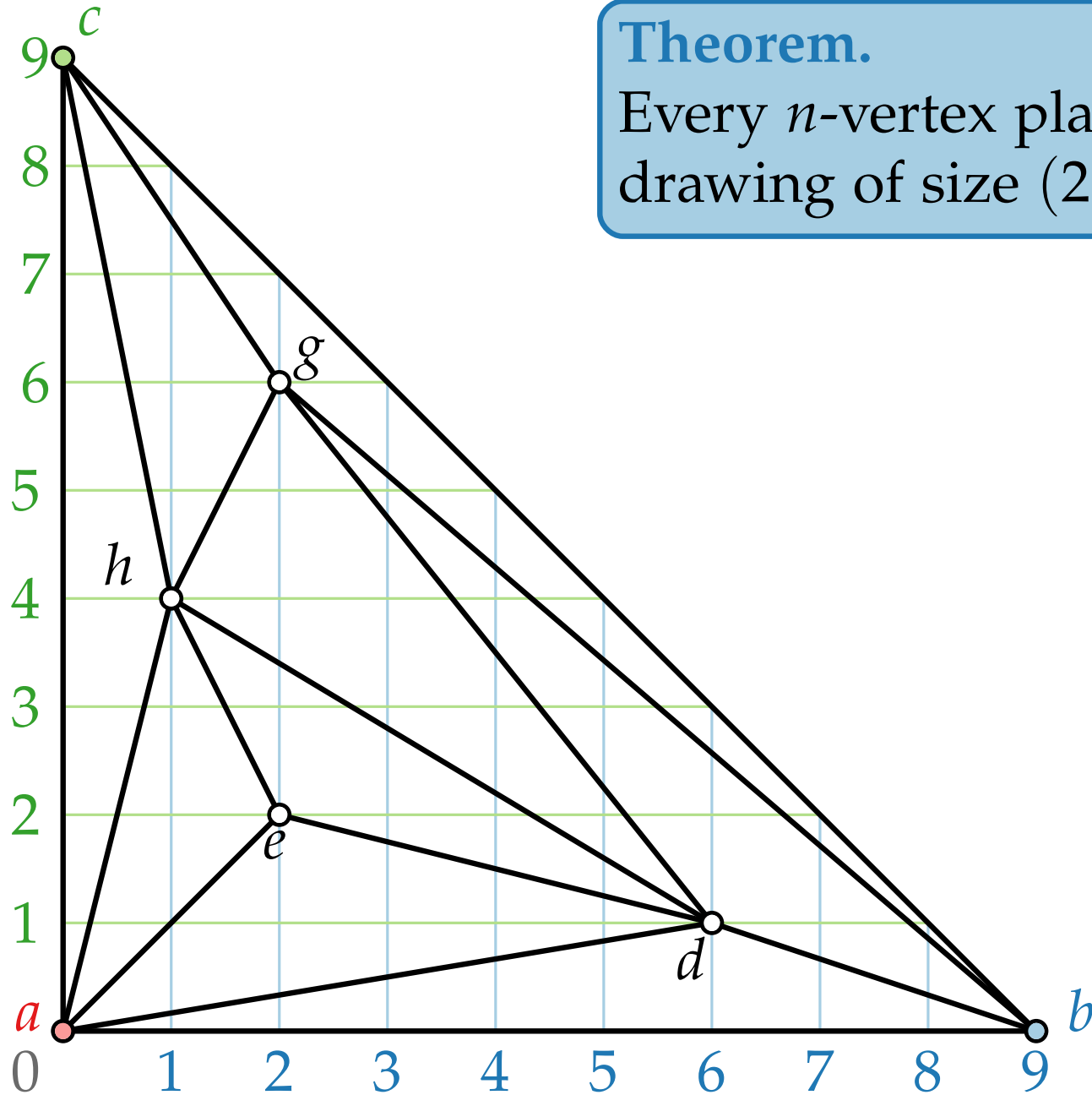
$$f(c) = (0, 0, 9) \quad f(h) = (4, 1, 4)$$

Schnyder Drawing – Example

Theorem.

Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 5) \times (2n - 5)$.

[Schnyder '89]



$$n = 7, 2n - 5 = 9 \quad f(d) = (2, 6, 1)$$

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Part IV:
Weak Barycentric Representation

Weak Barycentric Representation

A **weak barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to V :

$$\phi: V \rightarrow \mathbb{R}_{\geq 0}^3, v \mapsto (v_1, v_2, v_3)$$

with the following properties:

(W1) $v_1 + v_2 + v_3 = 1$ for all $v \in V$,

(W2) for each $xy \in E$ and each $z \in V \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with

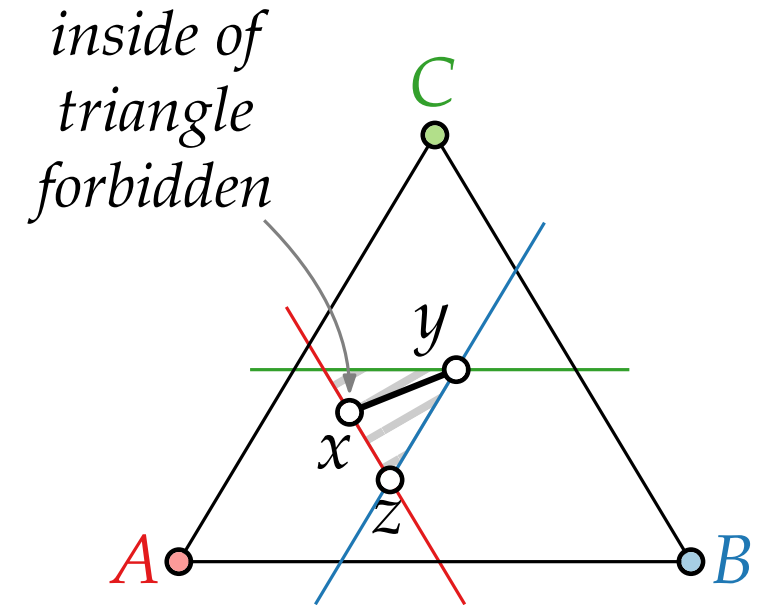
$$(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1}) \text{ and } (y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1}).$$

Lemma.

For a weak barycentric representation $\phi: v \mapsto (v_1, v_2, v_3)$ and a triangle A, B, C , the mapping

$$f: v \in V \mapsto v_1 A + v_2 B + v_3 C$$

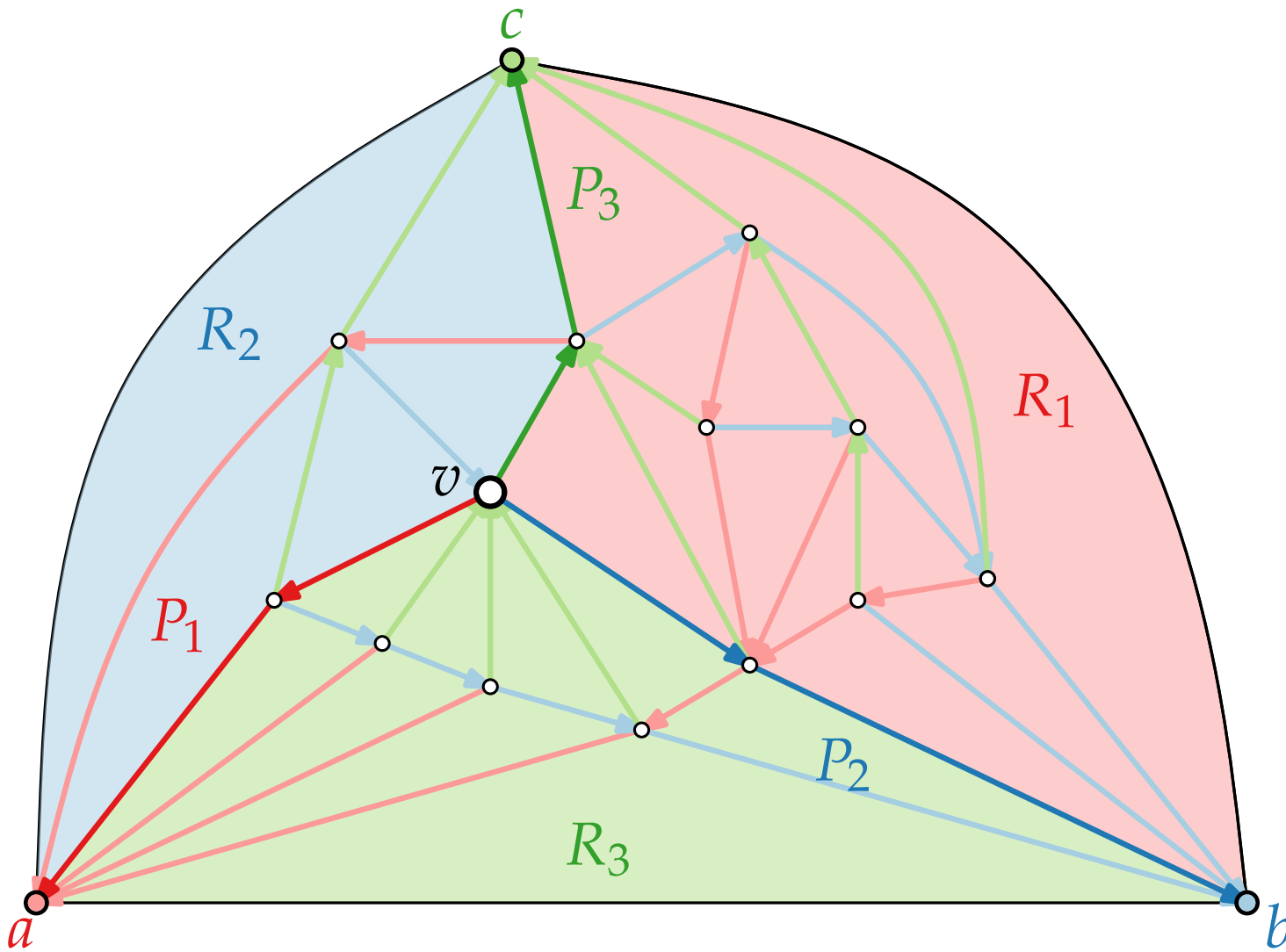
gives a **planar** drawing of G inside $\triangle ABC$.



i.e., either $y_k < z_k$ or $y_k = z_k$ and $y_{k+1} < z_{k+1}$

Proof as **exercise**.

Counting Vertices



$P_i(v)$: path from v to root of T_i .

$R_1(v)$: set of faces contained in P_2, bc, P_3 .

$R_2(v)$: set of faces contained in P_3, ca, P_1 .

$R_3(v)$: set of faces contained in P_1, ab, P_2 .

$$v_i = |V(R_i(v))| - |P_{i-1}(v)|$$

$$v_1 = 10 - 3 = 7$$

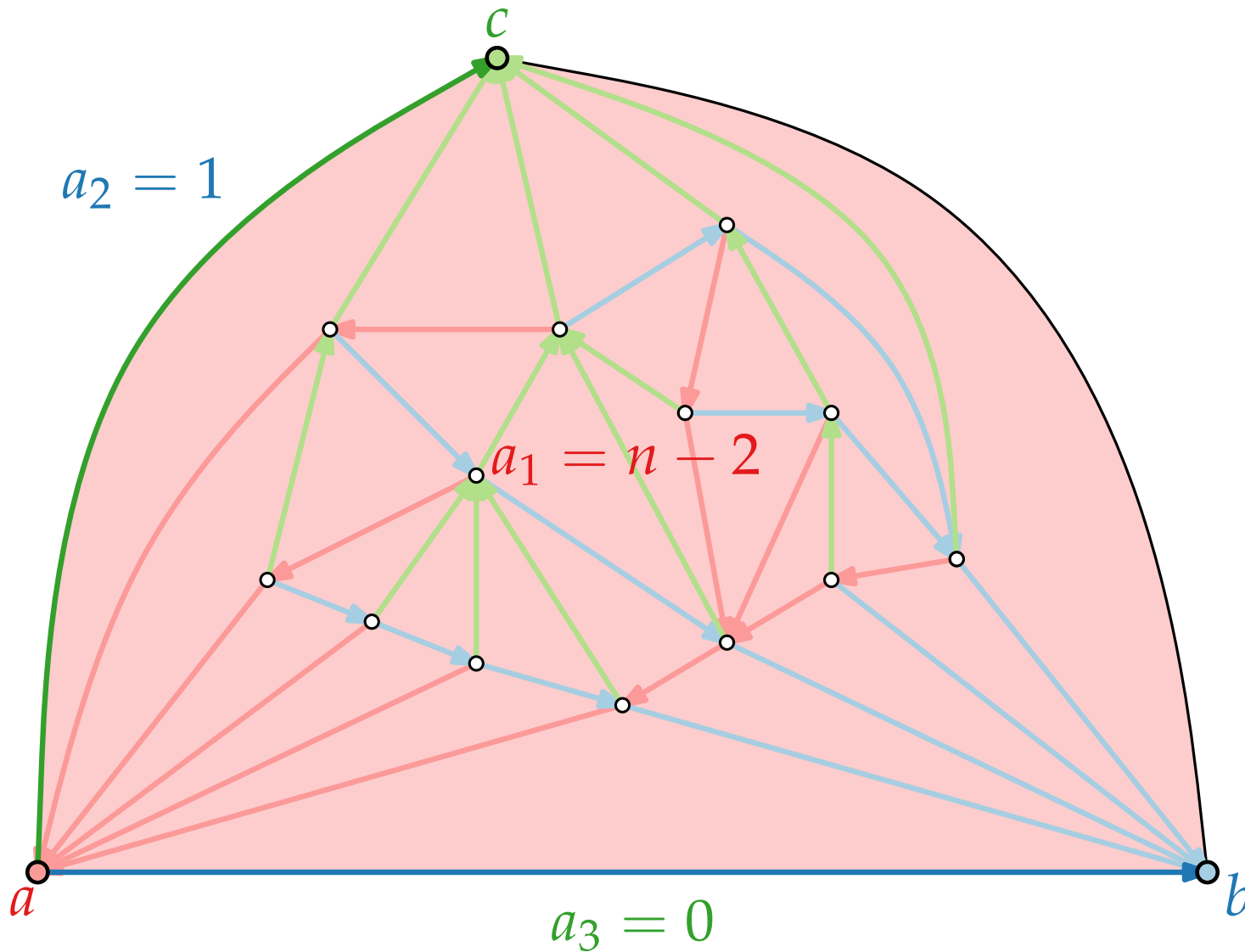
$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.
- $v_1 + v_2 + v_3 = n - 1$

Counting Vertices



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$$v_i = |V(R_i(v))| - |P_{i-1}(v)|$$

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.
- $v_1 + v_2 + v_3 = n - 1$

Schnyder Drawing^{*}

Set $A = (0, 0)$, $B = (n - 1, 0)$, and $C = (0, n - 1)$.

Theorem.

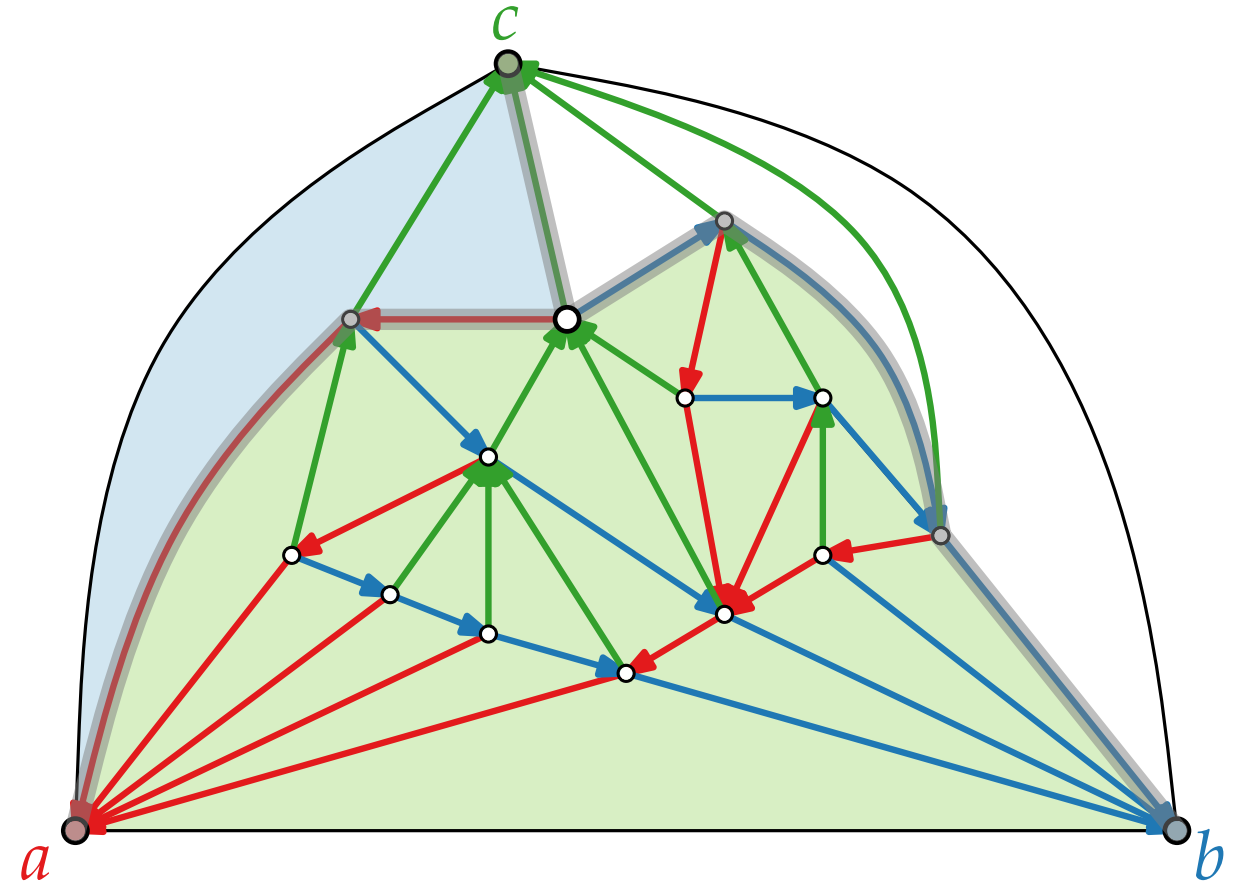
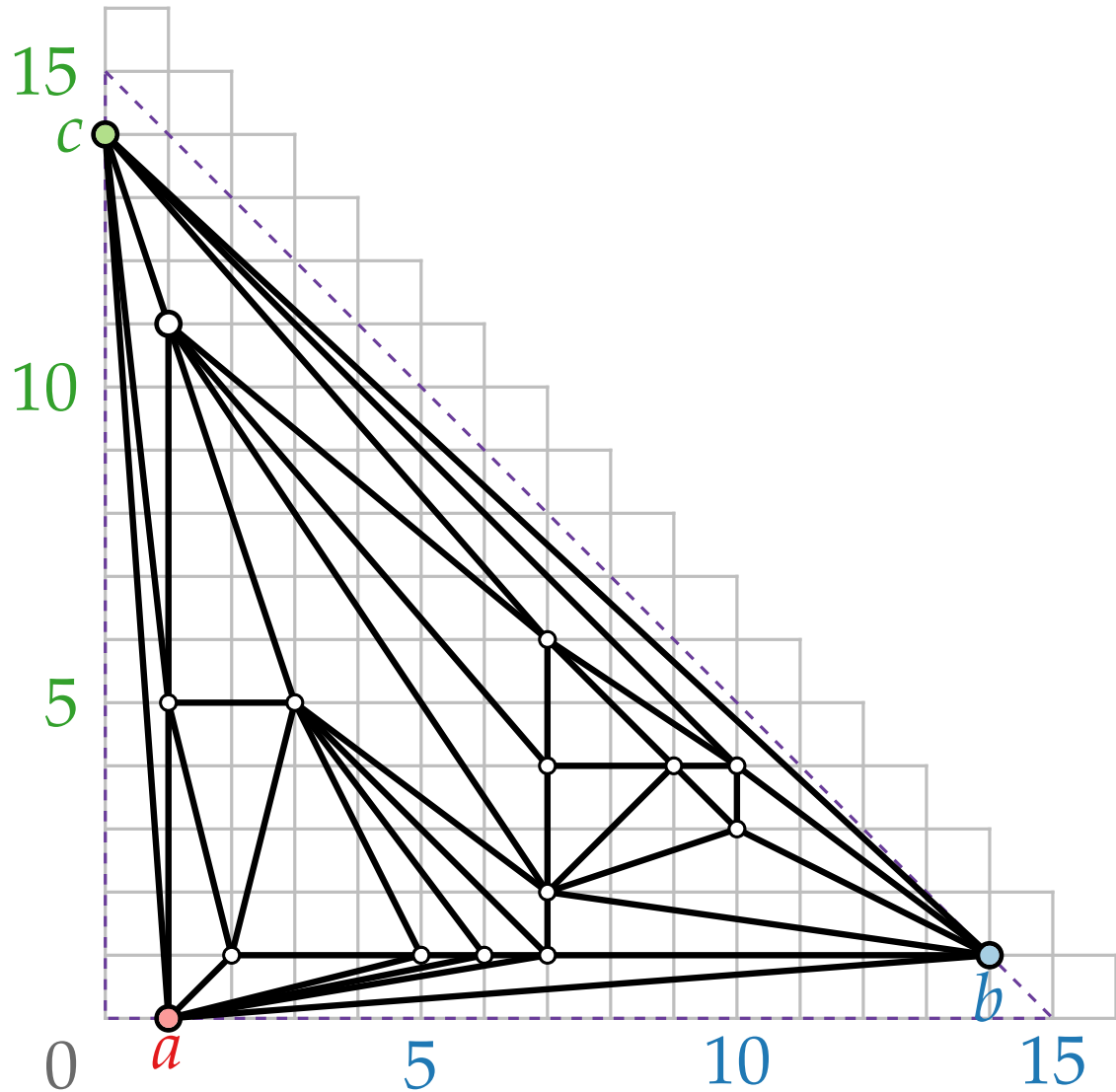
[Schnyder '90]

For a plane triangulation G , the mapping

$$f: v \mapsto \frac{1}{n-1} (v_1, v_2, v_3)$$

is a barycentric representation of G , which thus gives a planar straight-line drawing of G on the $(n - 2) \times (n - 2)$ grid.

Schnyder Drawing* – Example



$$n = 16, n - 2 = 14$$

$$f(a) = (n - 2, 1, 0)$$

$$f(b) = (0, n - 2, 1)$$

$$f(c) = (1, 0, n - 2)$$

Results & Variations

Theorem.

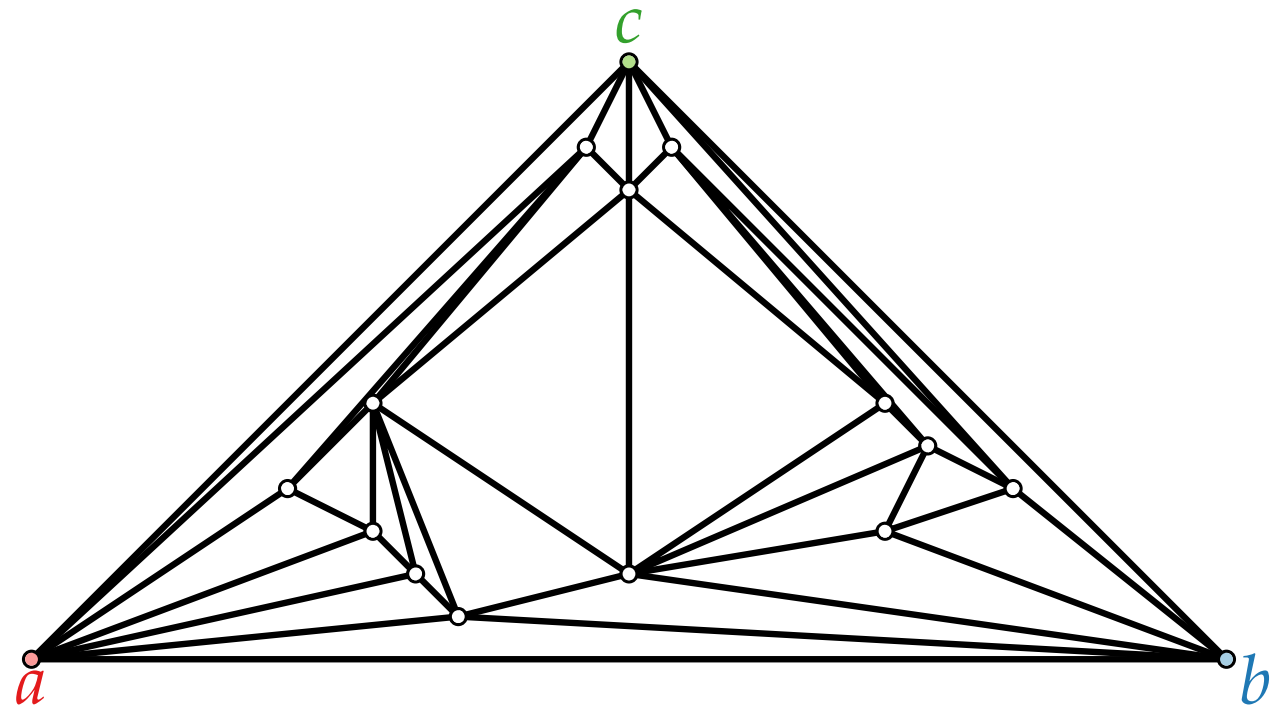
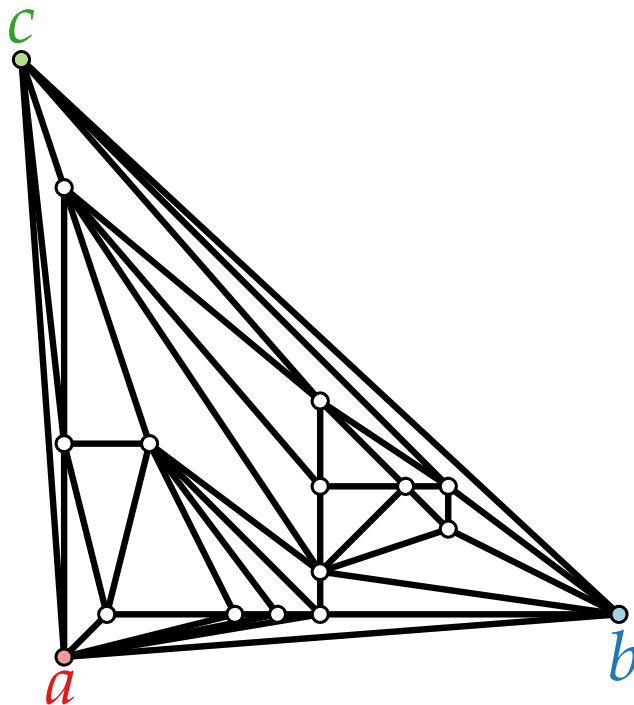
[De Fraysseix, Pach, Pollack '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

Theorem.

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Every n -vertex planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

Theorem.

[Chrobak & Kant '97]

Every n -vertex 3-connected planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$ where all faces are drawn convex. Such a drawing can be computed in $O(n)$ time.

Theorem.

[Felsner '01]

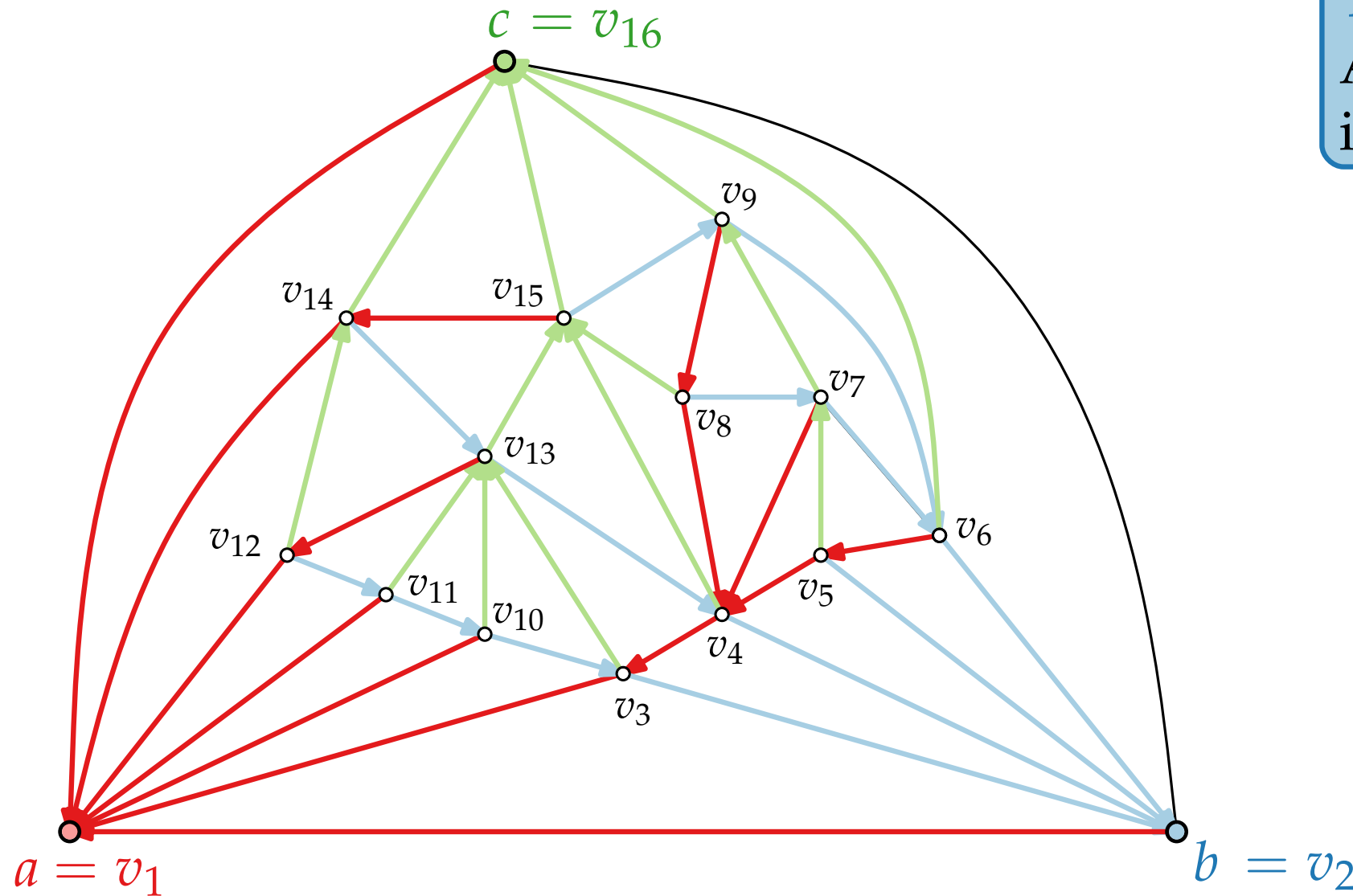
Every 3-connected planar graph with f faces has a planar straight-line drawing of size $(f - 1) \times (f - 1)$ where all faces are drawn convex. Such a drawing can be computed in $O(n)$ time.

Part V:
From Schnyder to Canonical Order
... and back again

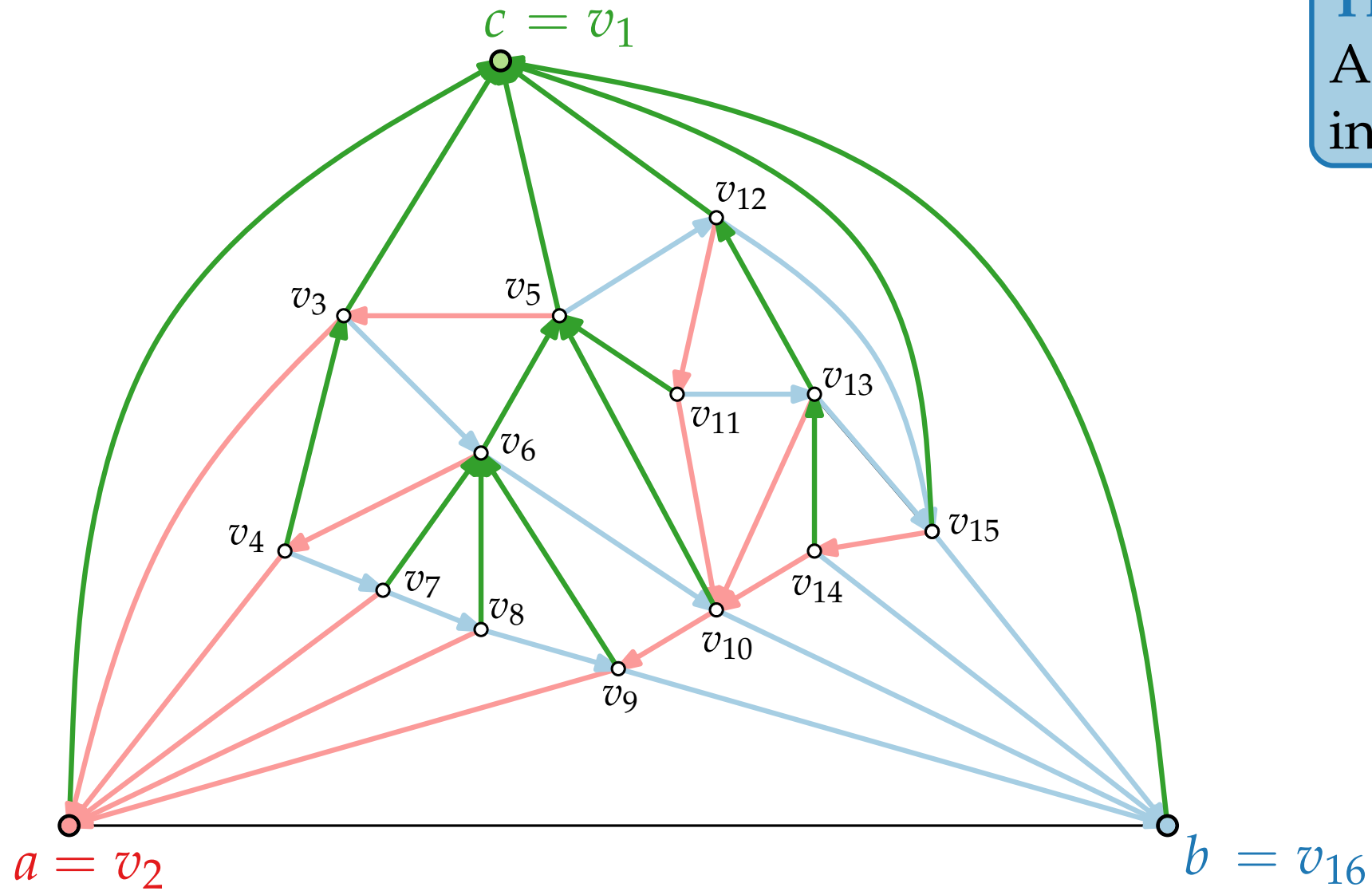
Schnyder Realizer \rightarrow Canonical Order

Theorem.

A ccw pre-order traversal on T_i induces a canonical order.



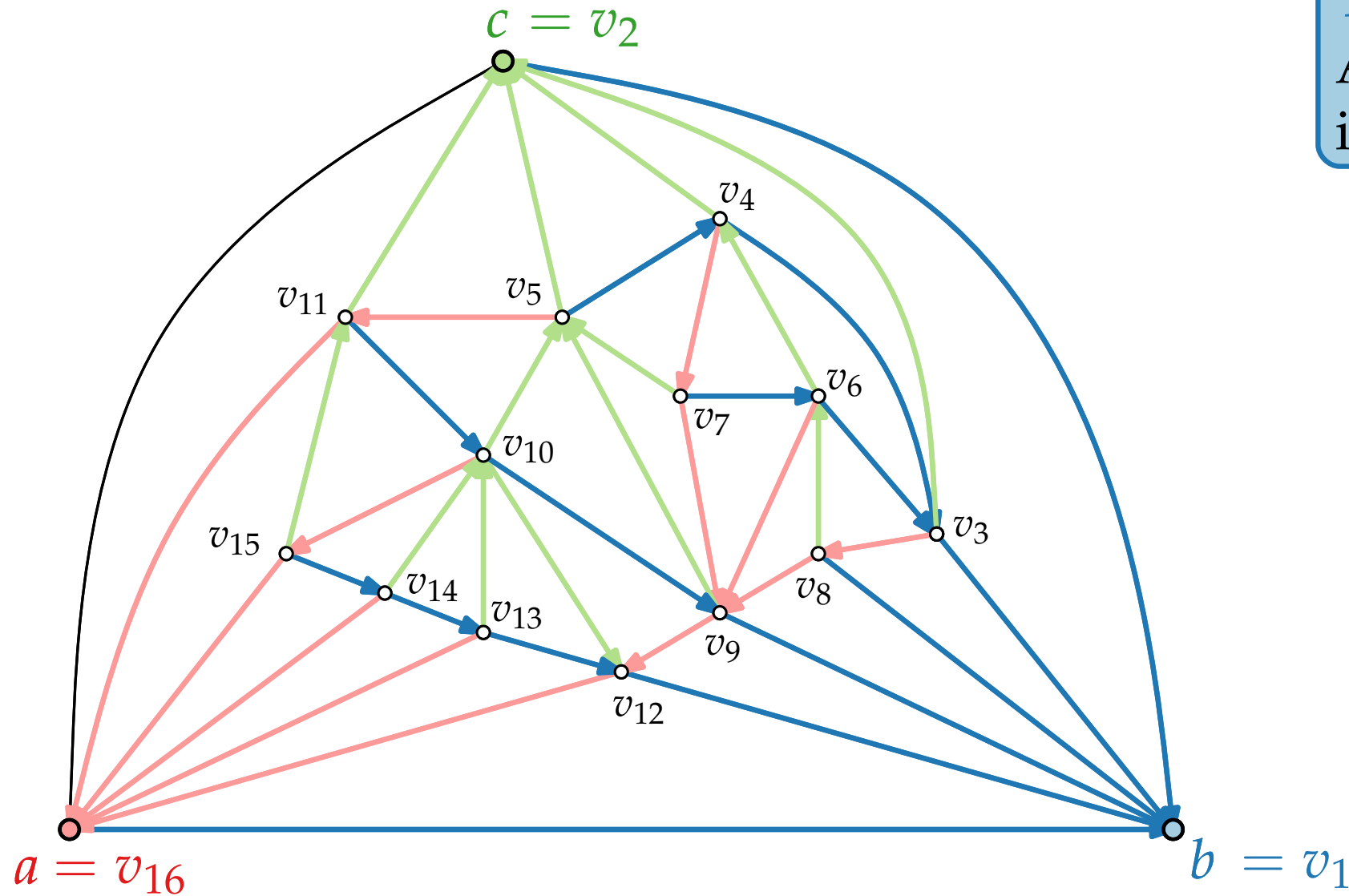
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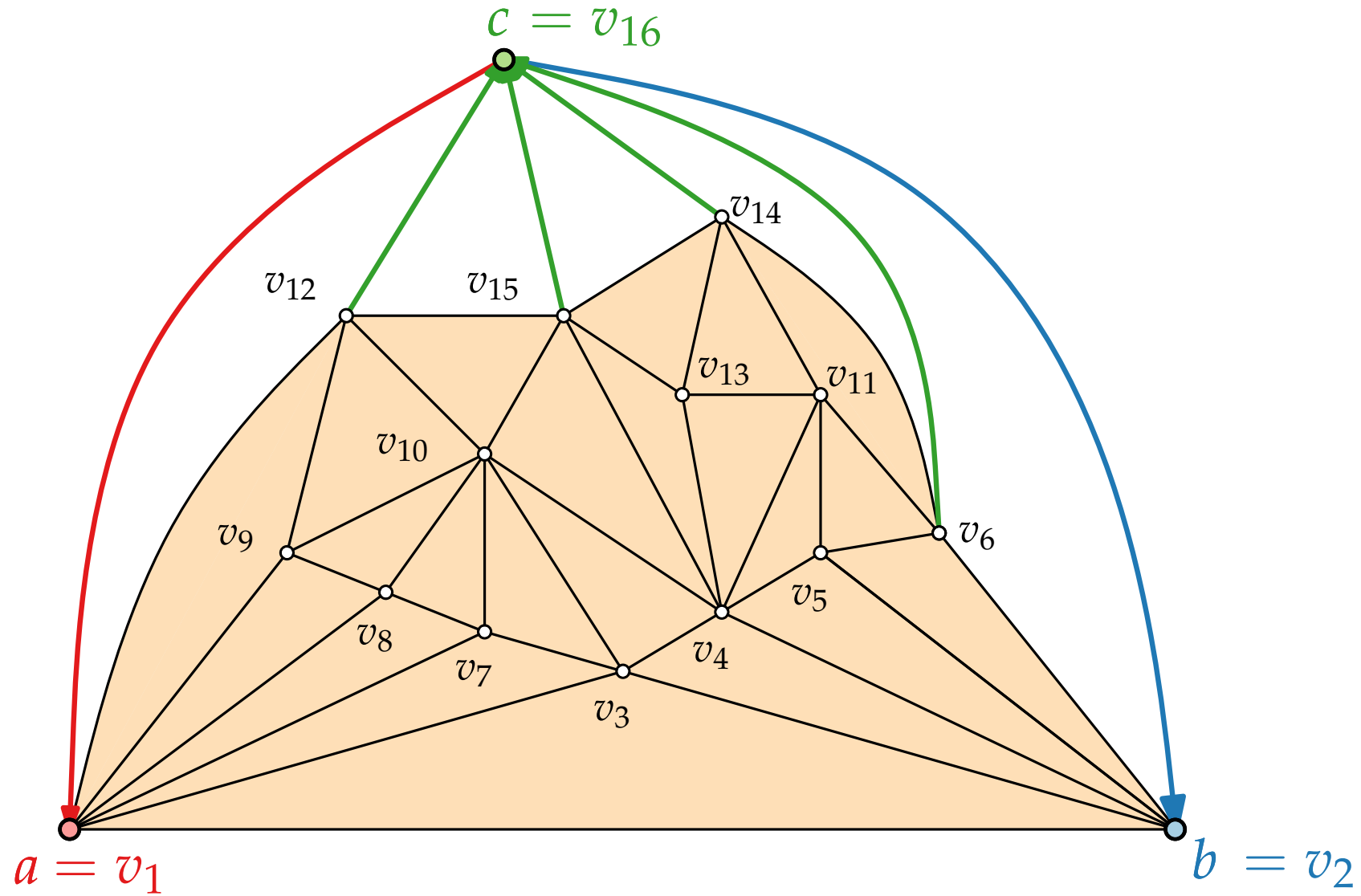
Schnyder Realizer \rightarrow Canonical Order



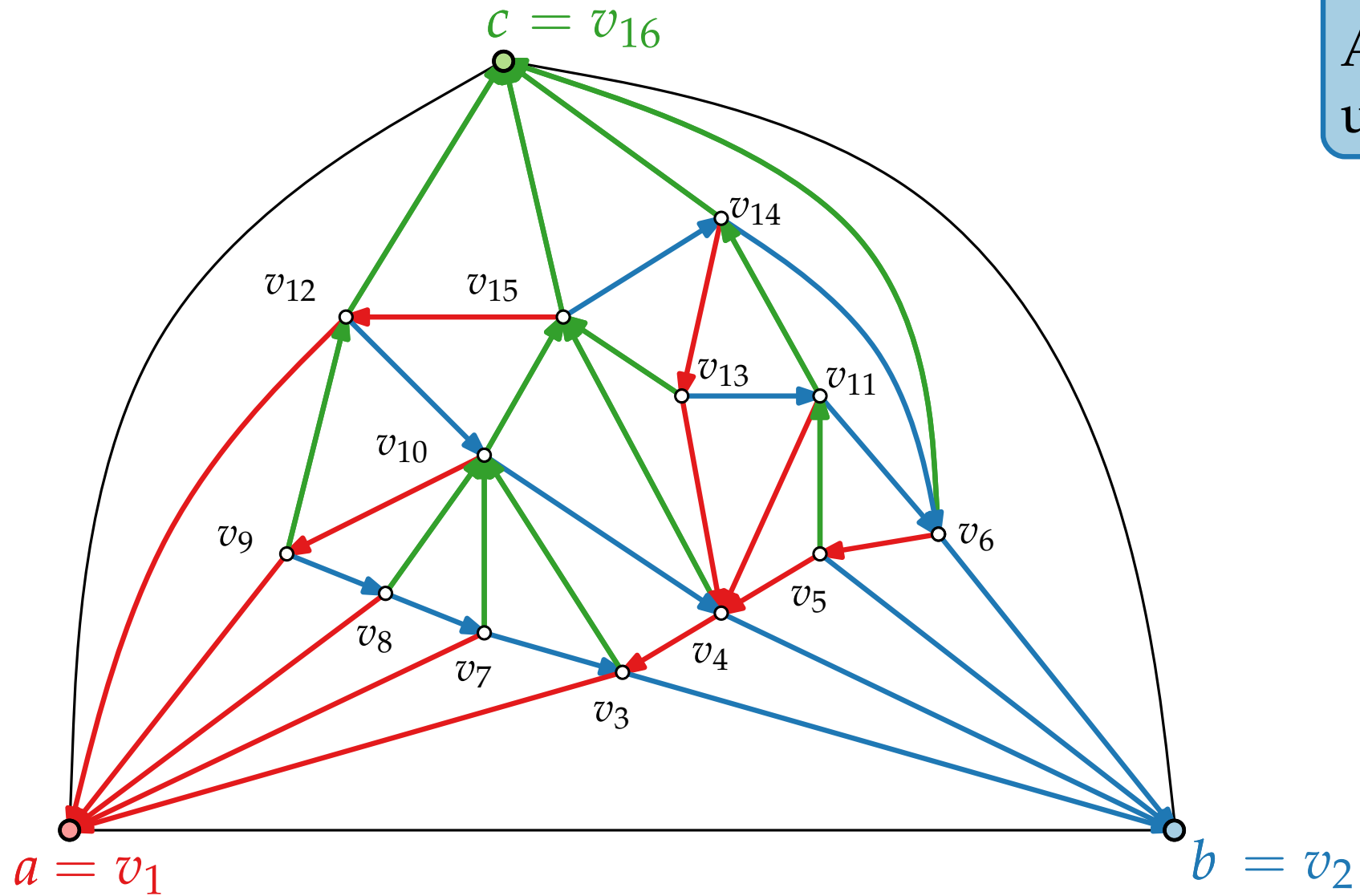
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Canonical Order \rightarrow Schnyder Realizer



Canonical Order \rightarrow Schnyder Realizer

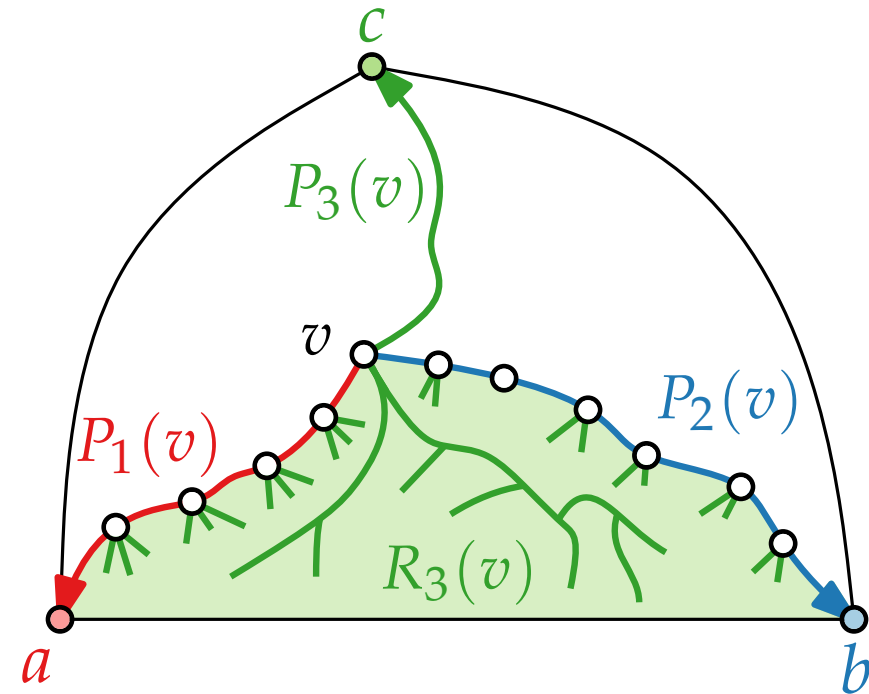


Theorem.

A canonical order induces a unique Schnyder Realizer.

Linear Time Computation

- Compute Canonical Order
- Compute Schnyder Realizer
- Goal: $v_i = |V(R_i(v))| - |P_{i-1}(v)|$
- With traversal of each T_i , compute:
 - The number of vertices in $P_i(v)$
 - The number of vertices in the subtree $T_i(v)$ of T_i rooted at v
- $|V(R_i(v))| = \sum_{u \in P_{i+1}(v)} |T_i(u)| + \sum_{u \in P_{i-1}(v)} |T_i(u)| - T_i(v)$
- Compute these sums in six tree traversals



Theorem.

[Schnyder '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.