Visualization of Graphs

Lecture 4:
Straight-Line Drawings of Planar Graphs I:
Canonical Ordering and Shift Method

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The slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, U. Trier

Part I:
Planar Straight-Line Drawings

Motivation

Why planar and straight-line?

[Bennett, Ryall, Spaltzeholz and Gooch '07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

Drawing conventions

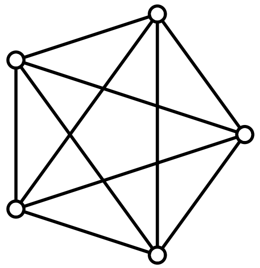
- No crossings ⇒ planar
- No bends ⇒ straight-line

Drawing aestethics

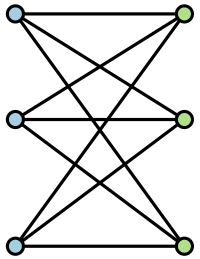
- Area

Planar Graphs

K_5



$K_{3,3}$



Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
neither K_5 nor $K_{3,3}$ minor of G

Characterization

Theorem. [Hopcroft & Tarjan 1974]
For a graph G with n vertices, there is an $\mathcal{O}(n)$ time algorithm to test whether G is planar.

Recognition

Also computes an embedding in $\mathcal{O}(n)$.

Theorem. [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has a planar drawing where the edges are straight-line segments.

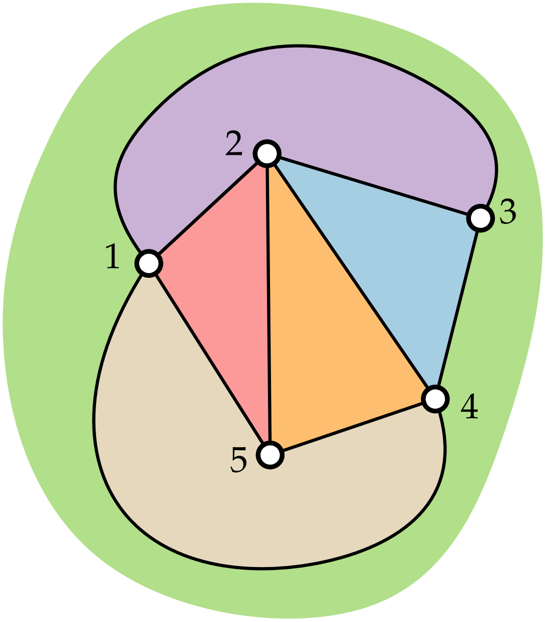
Drawing

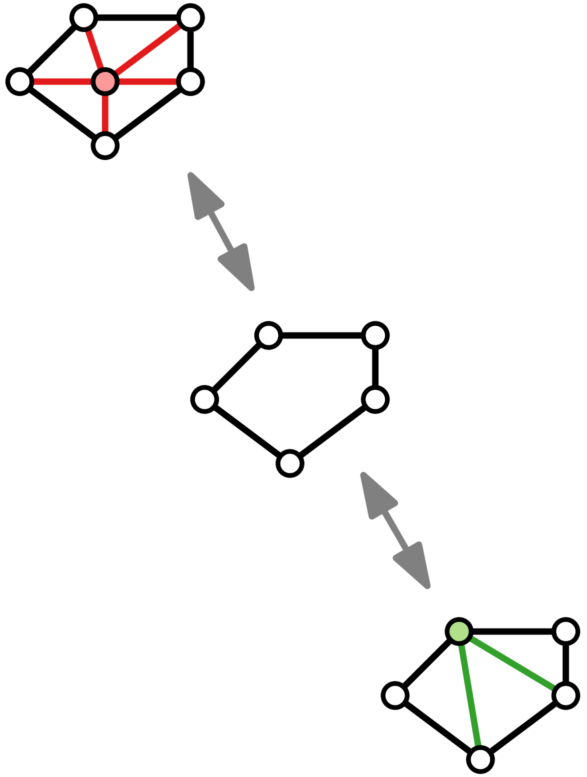
Triangulations

with planar embedding

A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.

A **maximal planar graph** is a planar graph where adding any edge would destroy planarity.





Observation.
A maximal plane graph is a plane triangulation.

Lemma.
A plane triangulation is at least 3-connected and thus has a unique planar embedding.

We focus on plane triangulations:

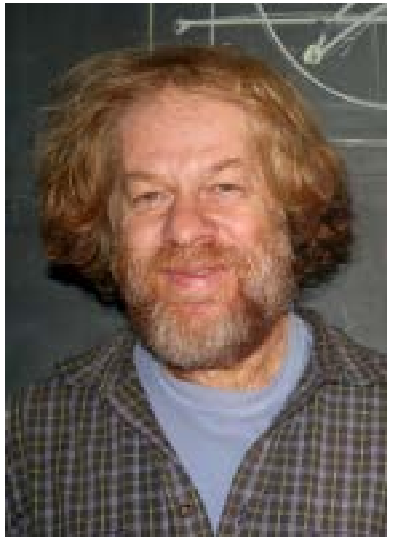
Lemma.
Every plane graph is subgraph of a plane triangulation.

Corollary.
Tutte's algorithm creates a planar straight-line drawing for every planar graph. (but with exponential area)

Planar Straight-Line Drawings

Hubert de Fraysseix
*Paris, France

János Pach
*1954, Budapest, Hungary



Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

Idea.

Richard Pollack
*1935, New York, USA
†2018, Montclair, USA

Theorem. [Schnyder '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$.

Part II:
Canonical Order

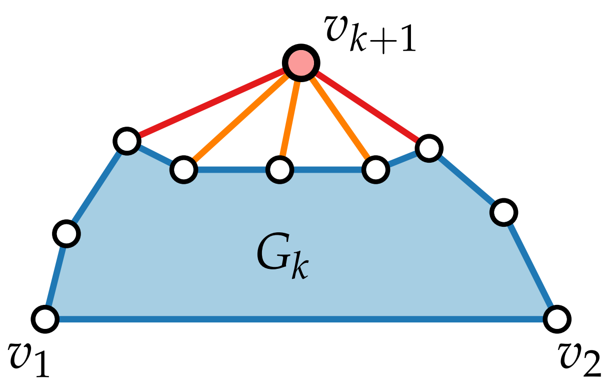
Canonical Order – Definition

Definition.
Let $G = (V, E)$ be a triangulated plane graph on $n \geq 3$ vertices.
An order $\pi = (v_1, v_2, \ldots, v_n)$ is called a **canonical order**, if the following conditions hold for each k, $3 \leq k \leq n$:

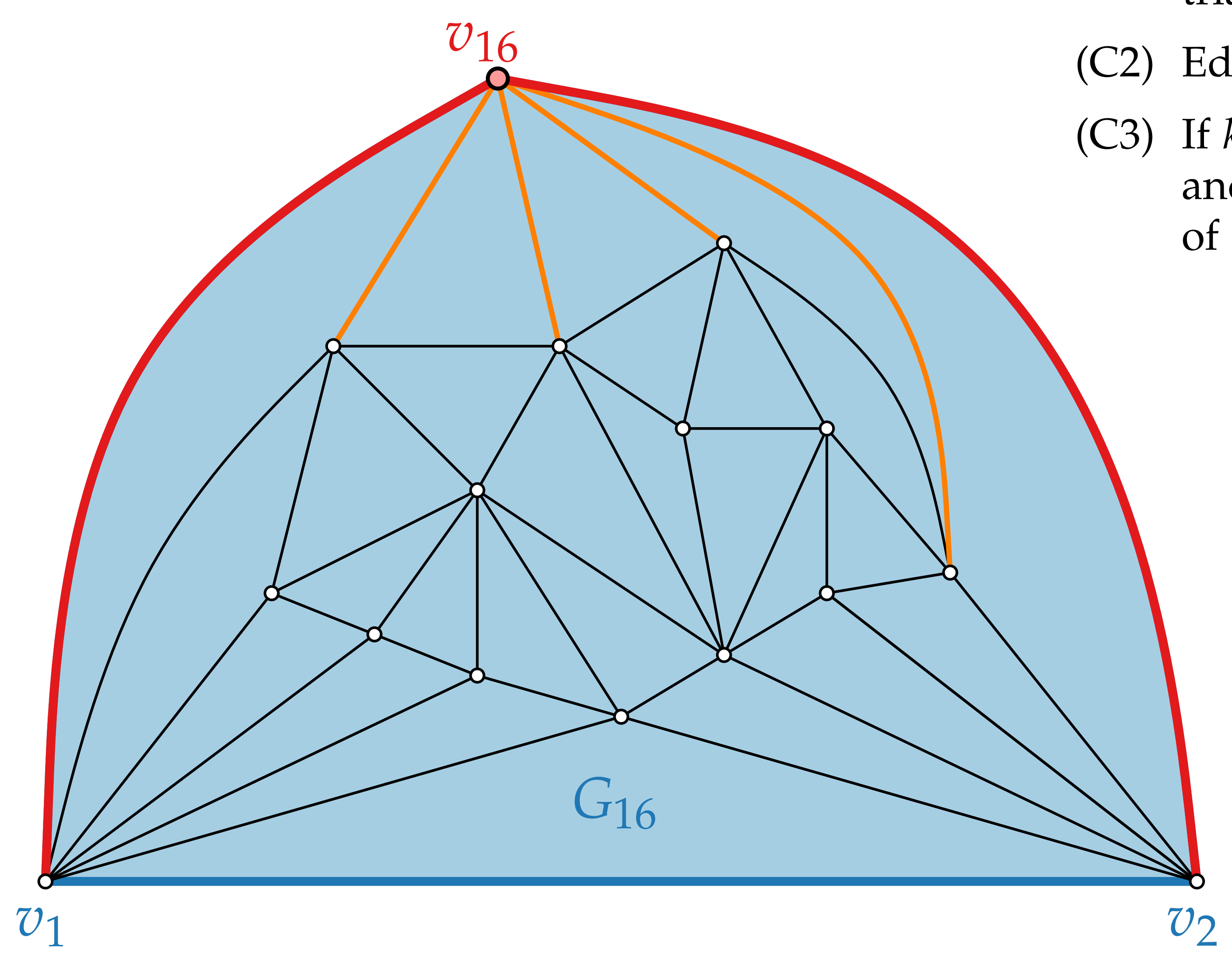
(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it G_k.

(C2) Edge (v_1, v_2) belongs to the outer face of G_k.

(C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k, and all neighbors of v_{k+1} in G_k appear on the boundary of G_k consecutively.



Canonical Order – Example



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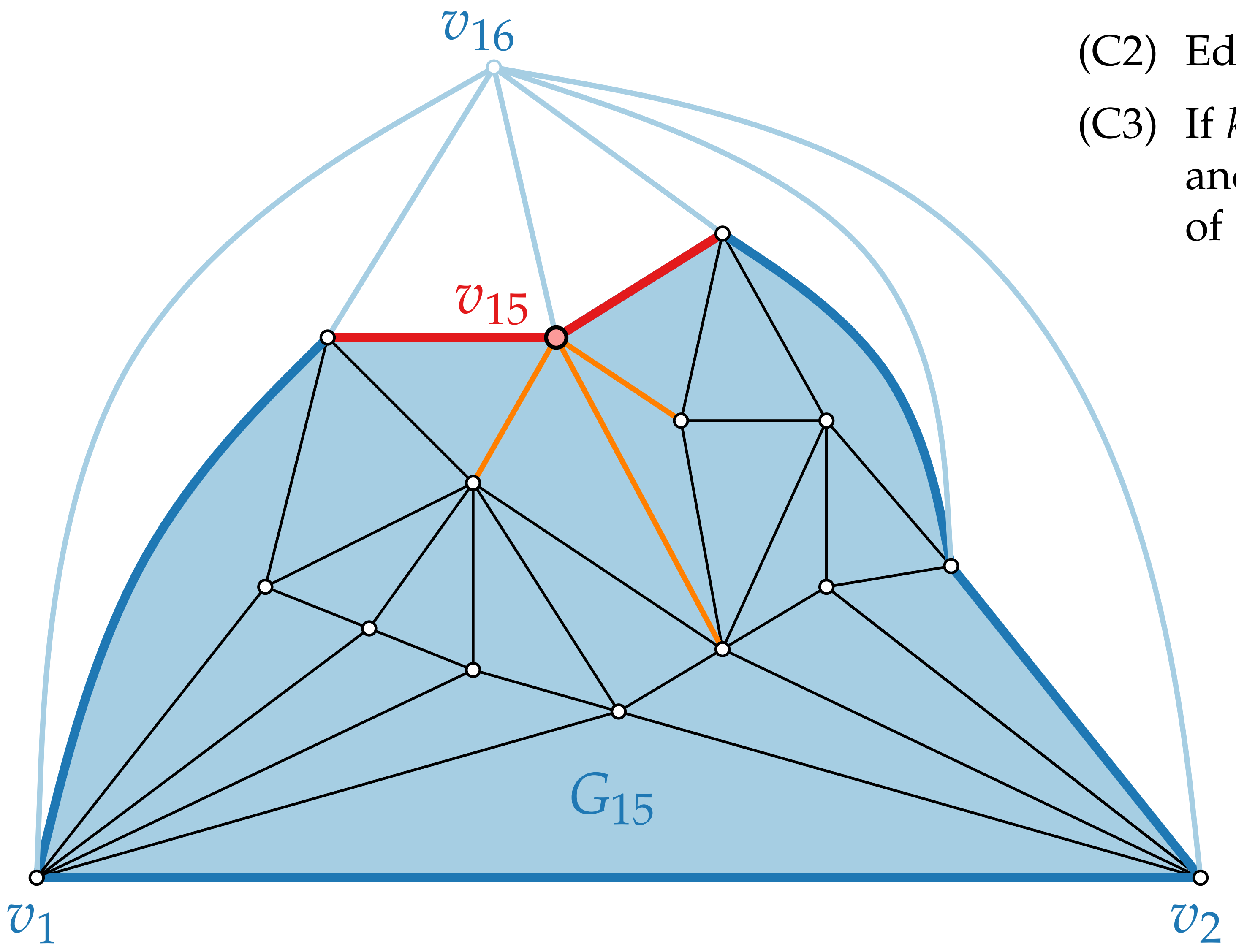

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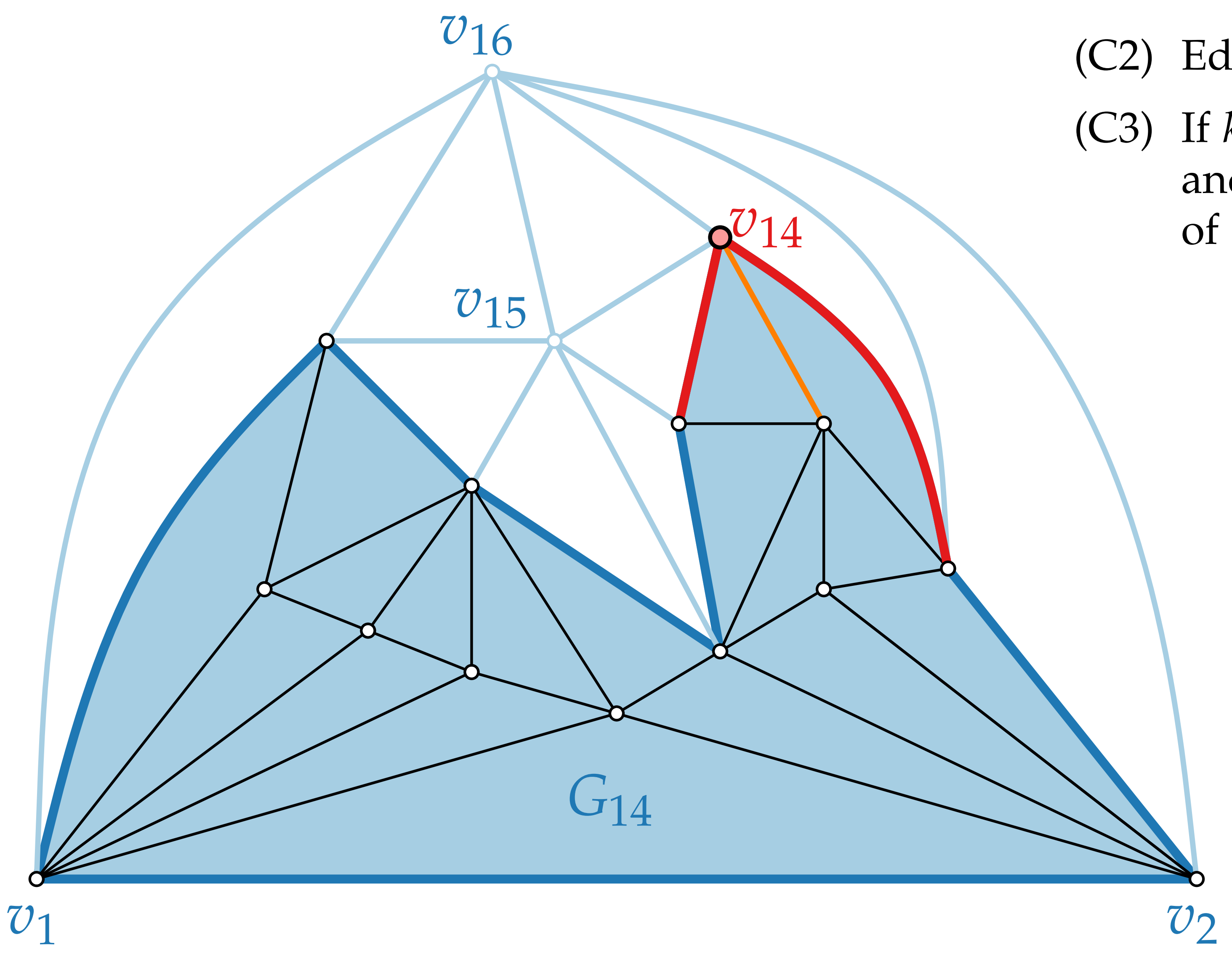

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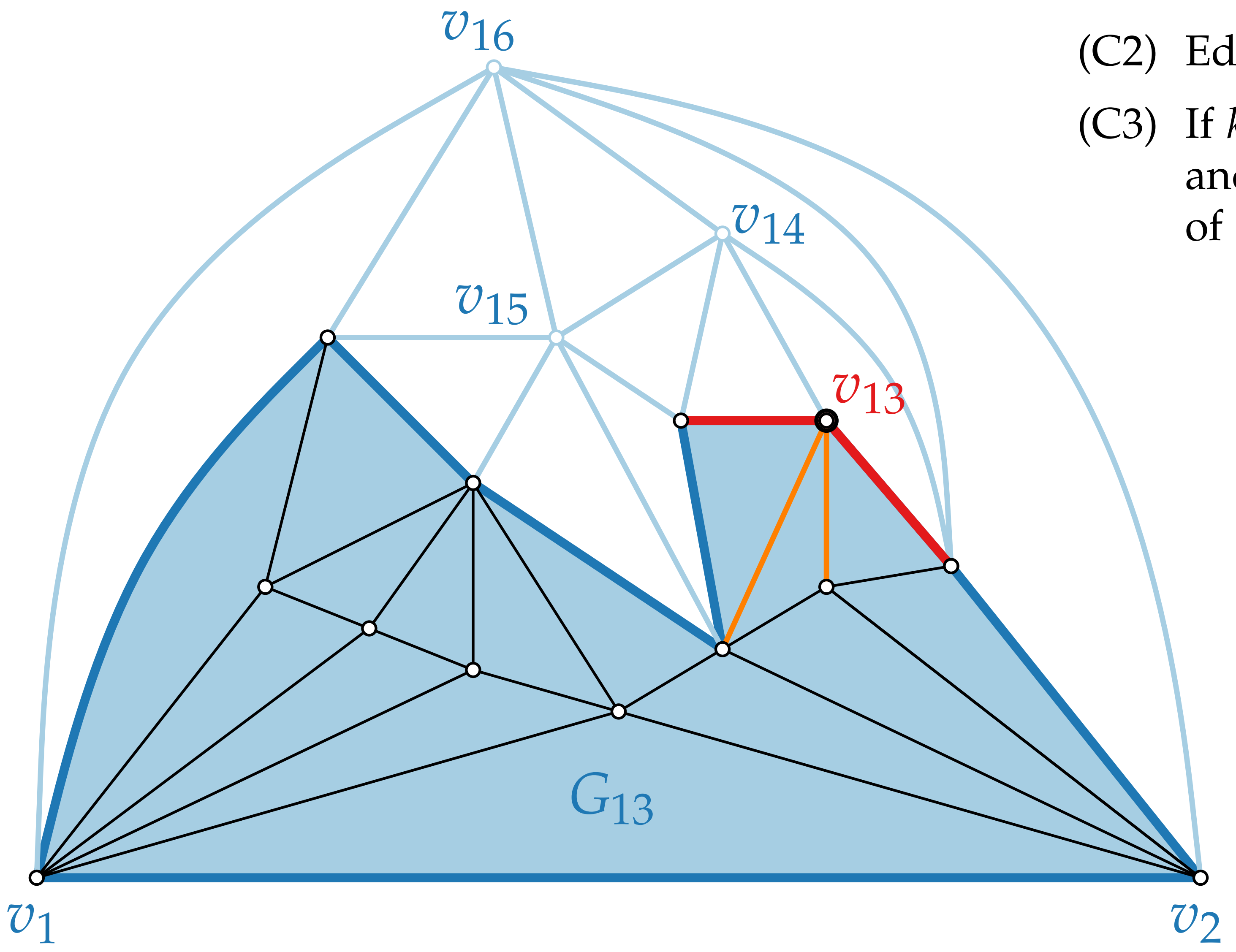

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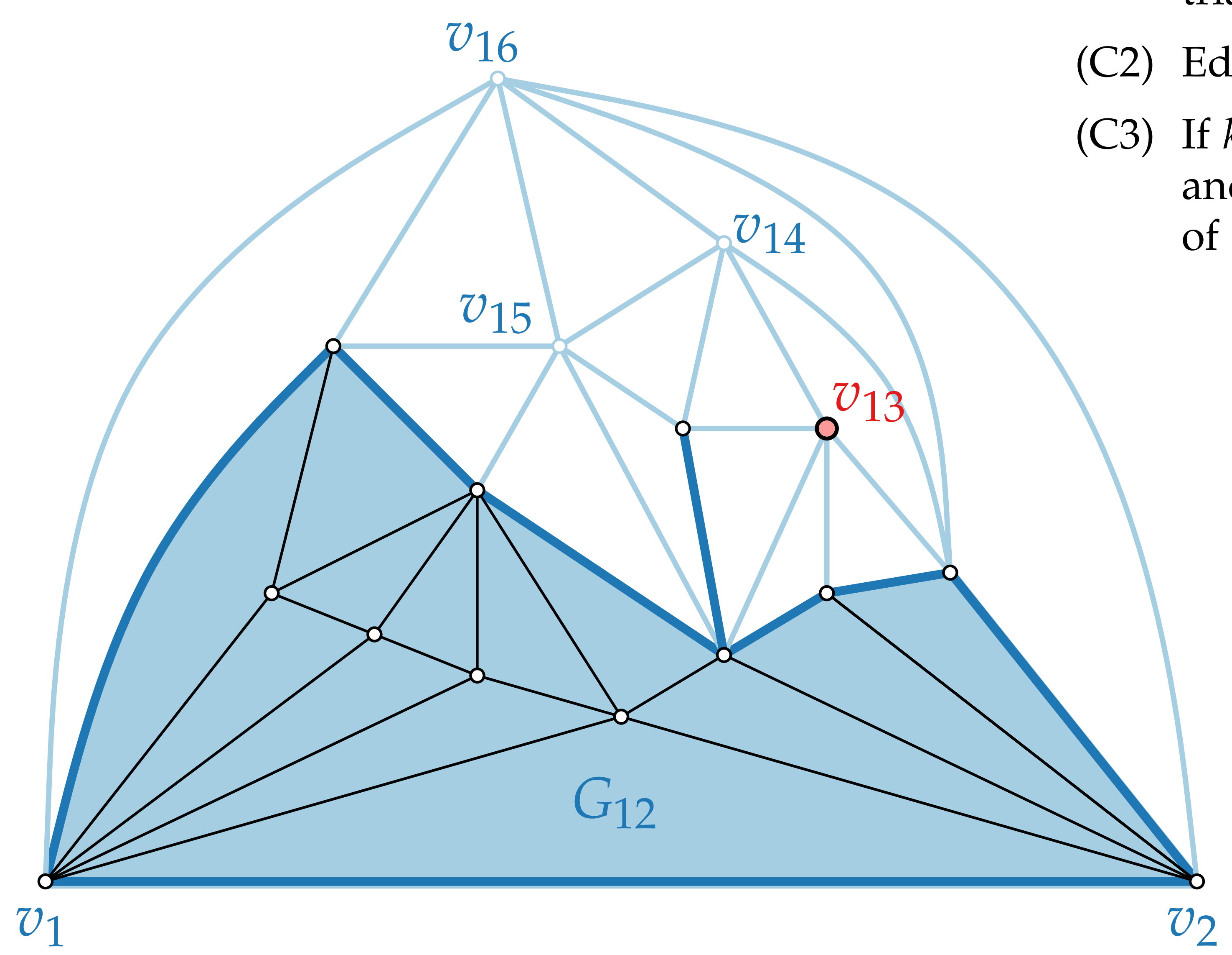

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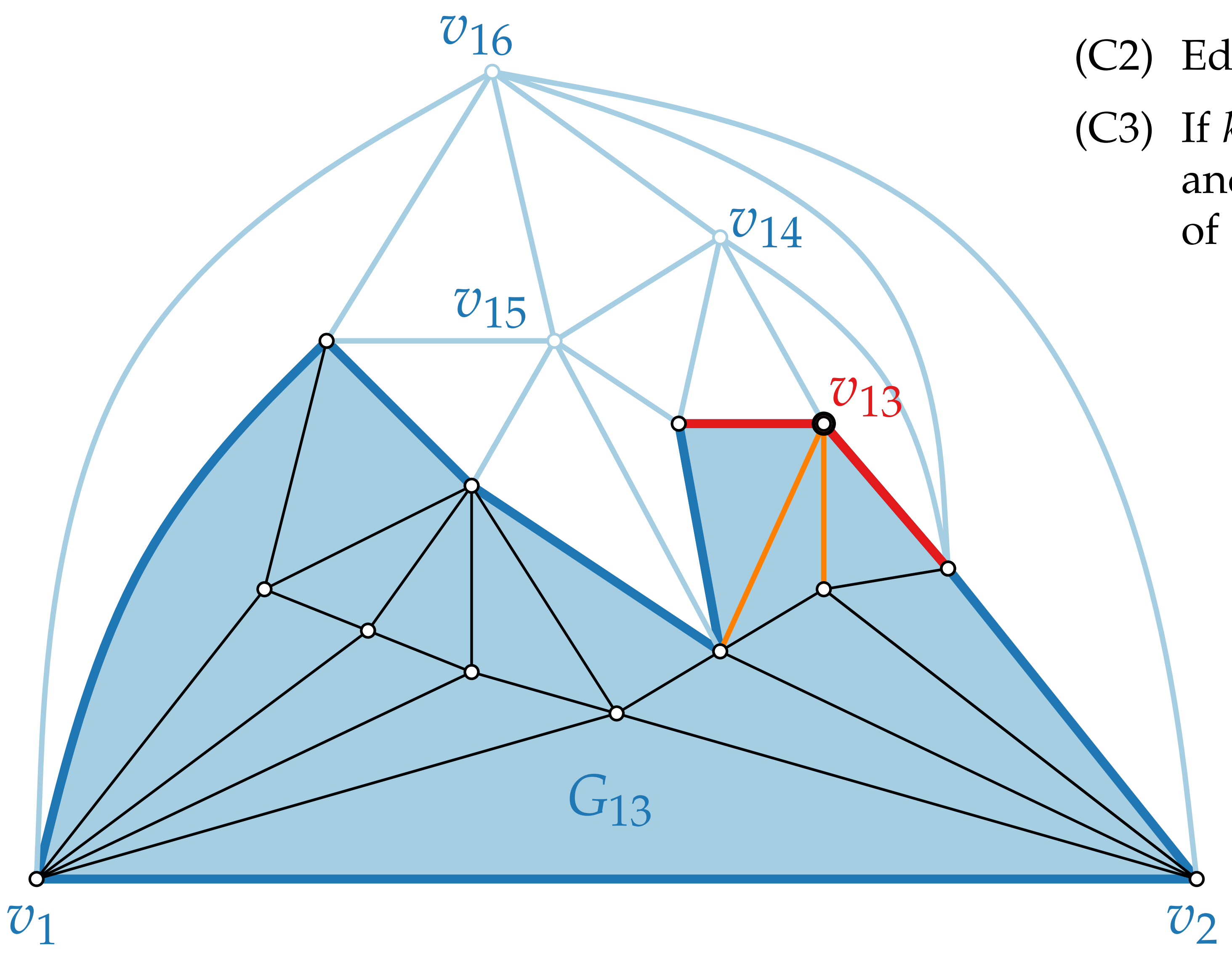

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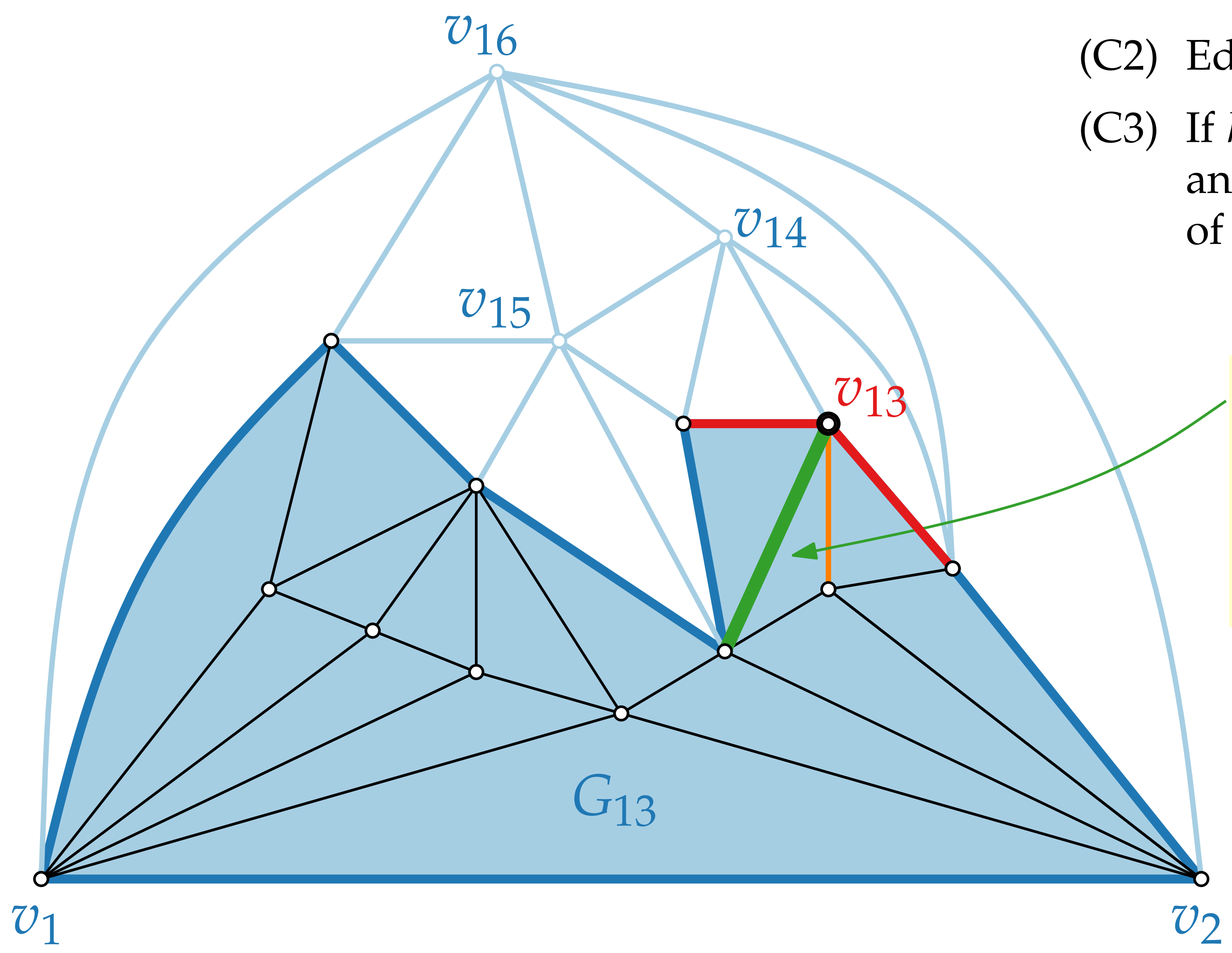

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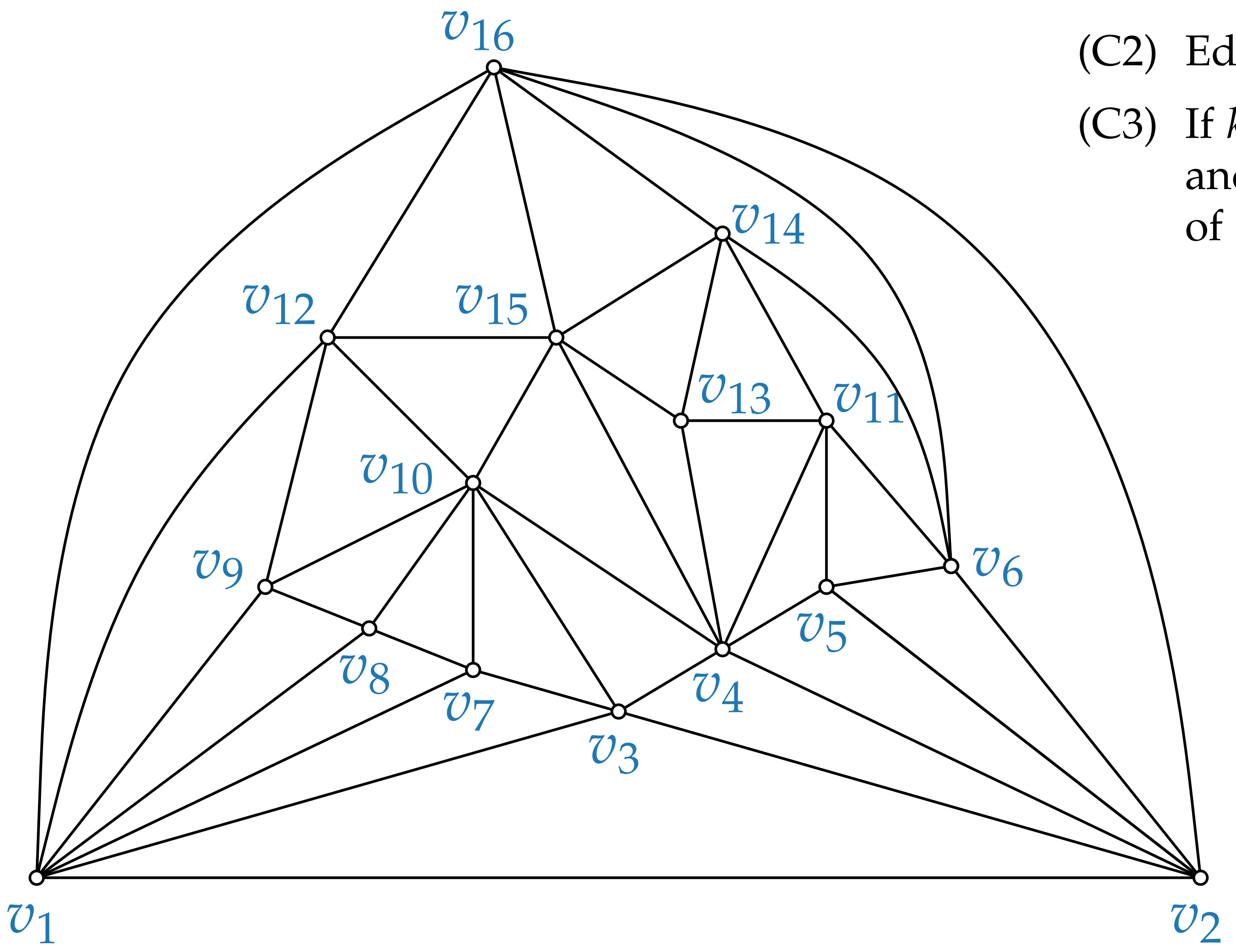
chord
edge joining two nonadjacent vertices in a cycle

Canonical Order – Example



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Part III:
Canonical Order – Existence

Canonical Order – Existence

(C1) G_k biconnected and internally triangulated

(C2) (v_1, v_2) on outer face of G_k

(C3) $k < n \Rightarrow v_{k+1}$ in outer face of G_k, neighbors of v_{k+1} in G_k consecutive on boundary

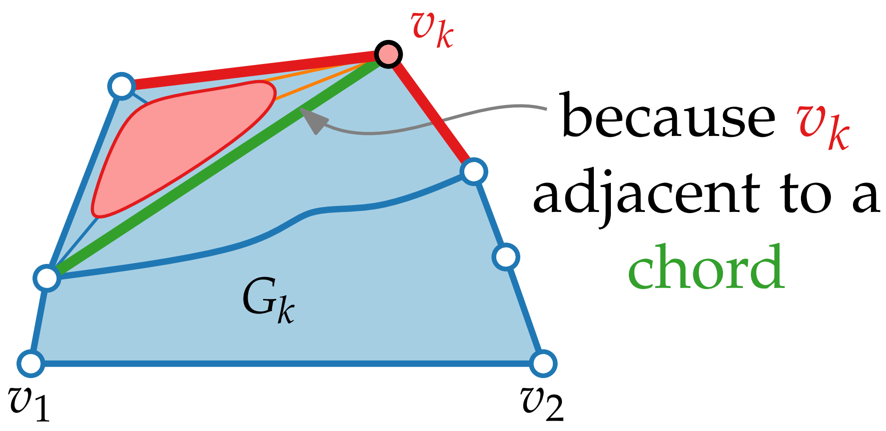
Lemma.
Every triangulated plane graph has a canonical order.

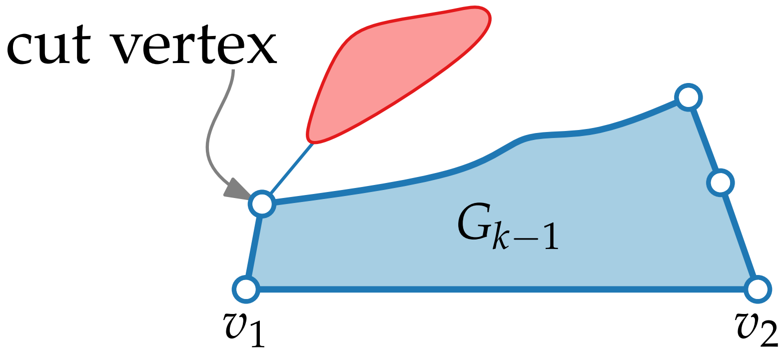
Base Case:
Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n. Conditions (C1) – (C3) hold.

Have to show:

Induction hypothesis:
Vertices $v_{n-1}, \ldots, v_{k+1}$ have been chosen such that conditions (C1) – (C3) hold for $k + 1 \leq i \leq n$.

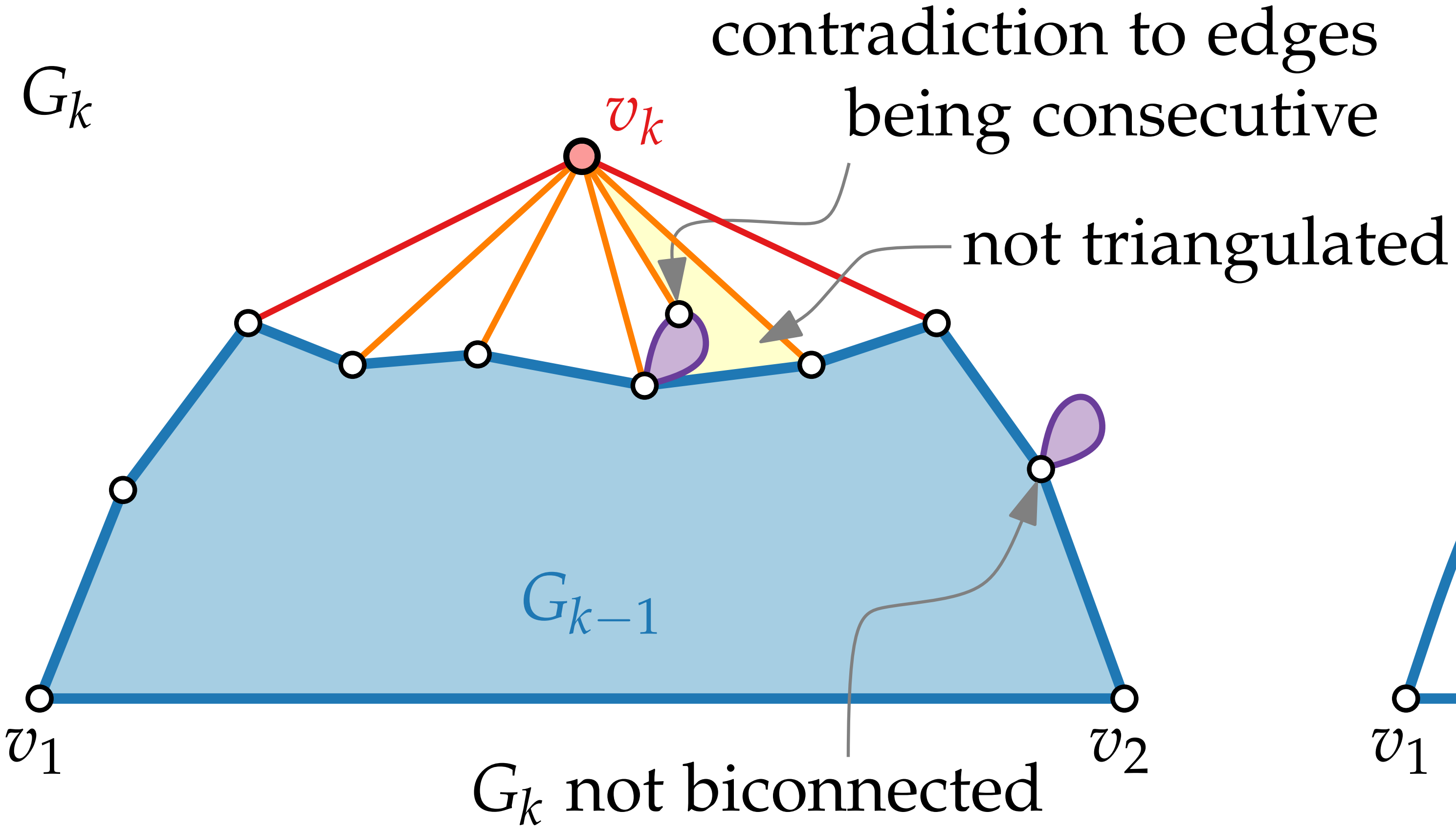
Induction step: Consider G_k. We search for v_k.



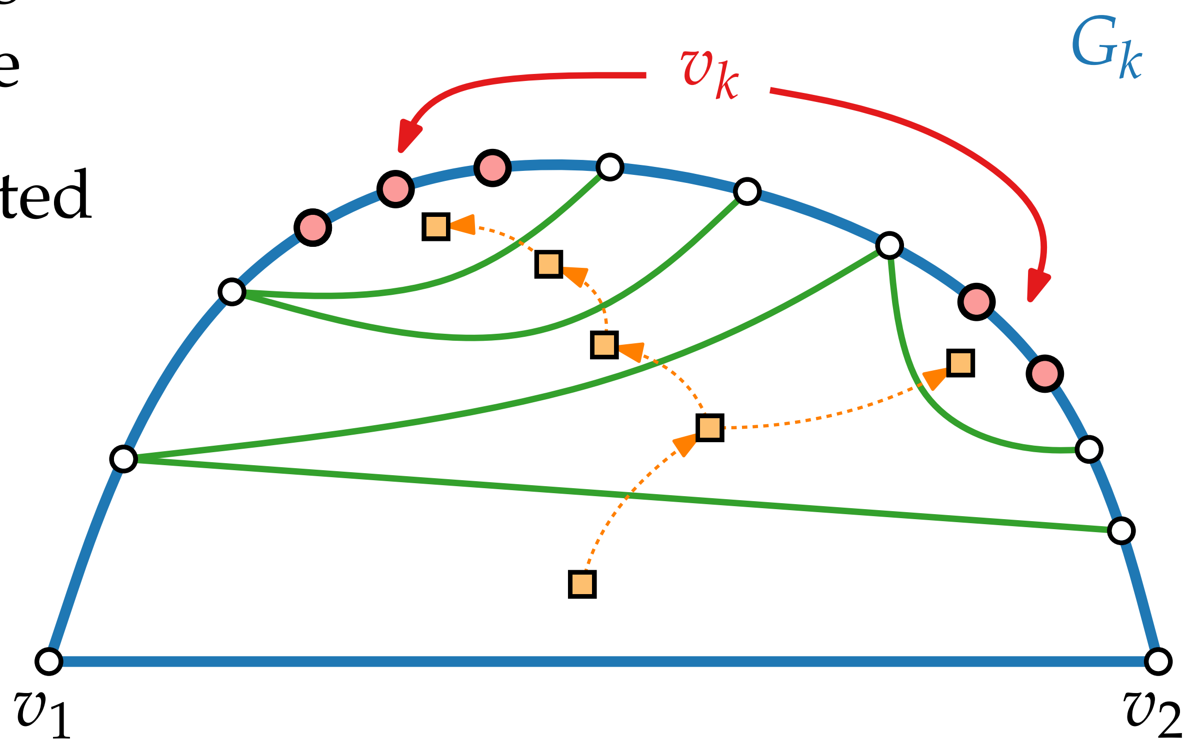


Canonical Order – Existence

Claim 1.
If v_k is not adjacent to a chord, then G_{k-1} is biconnected.



Claim 2.
There exists a vertex in G_k that is not adjacent to a chord as choice for v_k.



This completes proof of Lemma. □

Canonical Order – Implementation

outer face

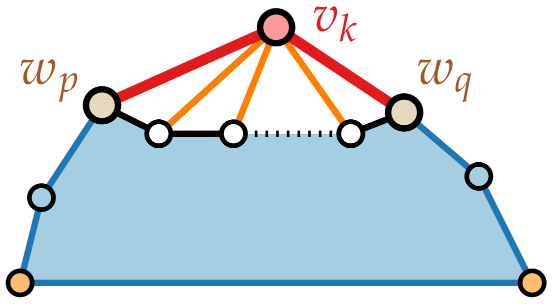
CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false

mark(v_1), mark(v_2), out(v_1), out(v_2), out(v_n) $\leftarrow$ true

for $k = n$ **to** 3 **do**
  choose v such that mark(v) = false, out(v) = true, and chords(v) = 0
  $v_k \leftarrow v$; mark(v) $\leftarrow$ true
  // Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{k-1} in G_{k-1} and let $w_p, \ldots, w_q$ be the neighbors of v_k
  out(w_i) $\leftarrow$ true for all $p < i < q$
  update number of chords for w_i and its neighbours

- chord(v): # chords adjacent to v
- out(v) = true iff v is currently outer vertex
- mark(v) = true iff v has received its number



Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

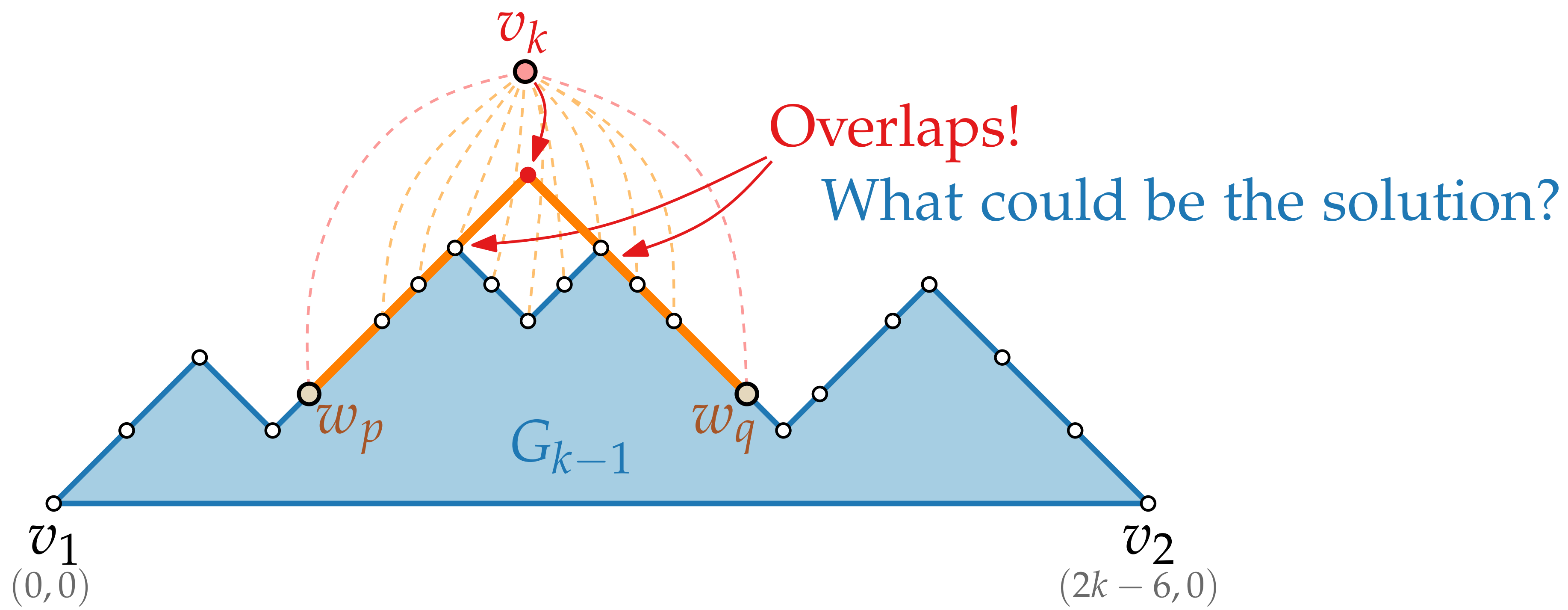
Part IV:
Shift Method

Shift Method – Idea

Drawing invariants:

G_{k-1} is drawn such that

- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$,
- boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone,
- each edge of the boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn with slopes ± 1.



Shift Method – Idea

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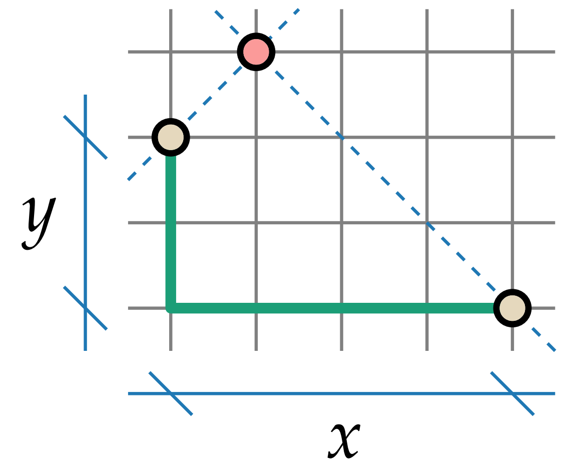
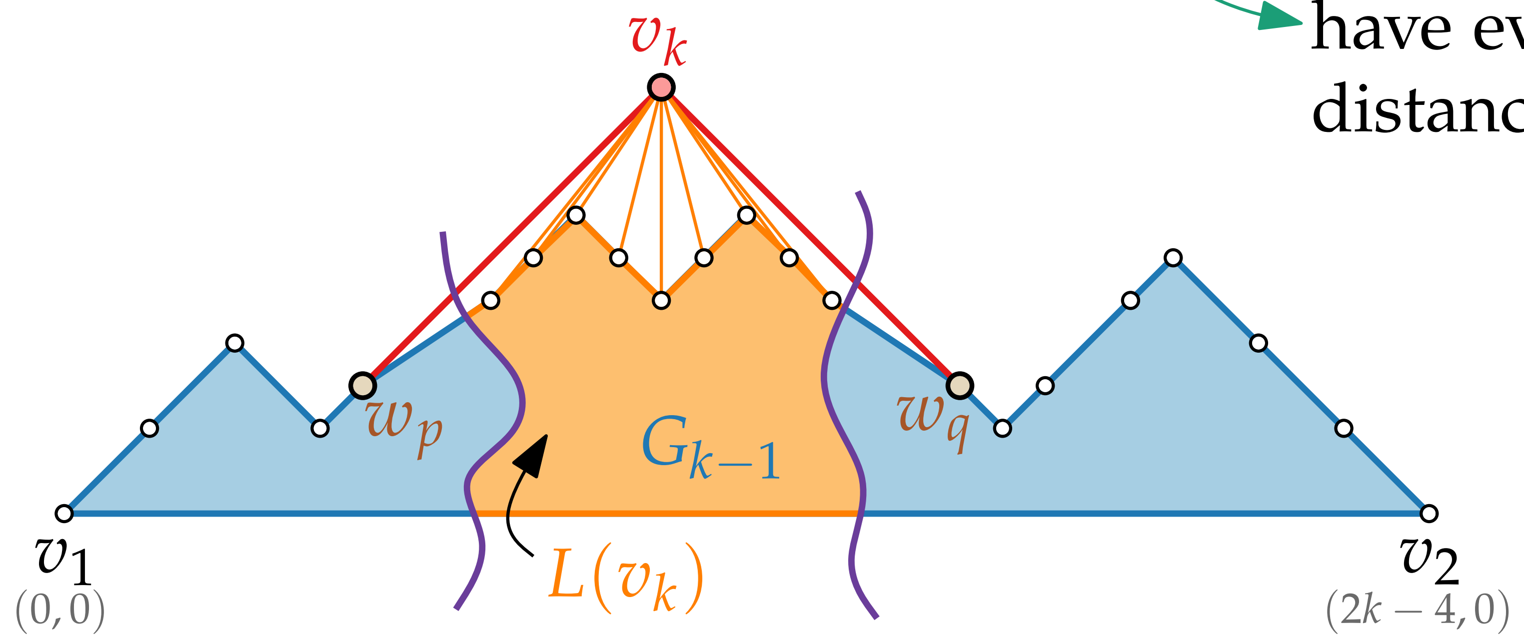
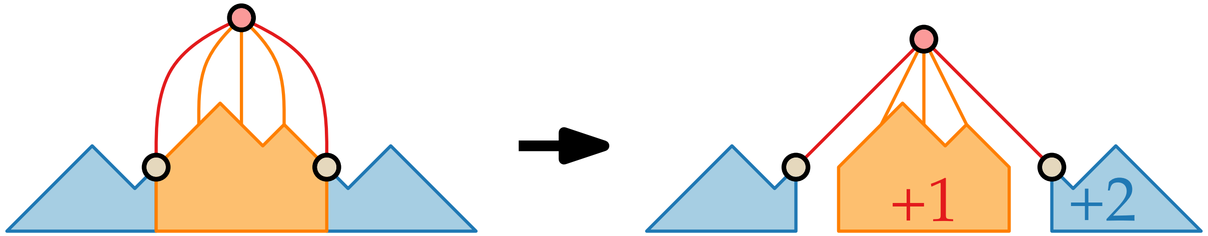
Does v_k land on grid?

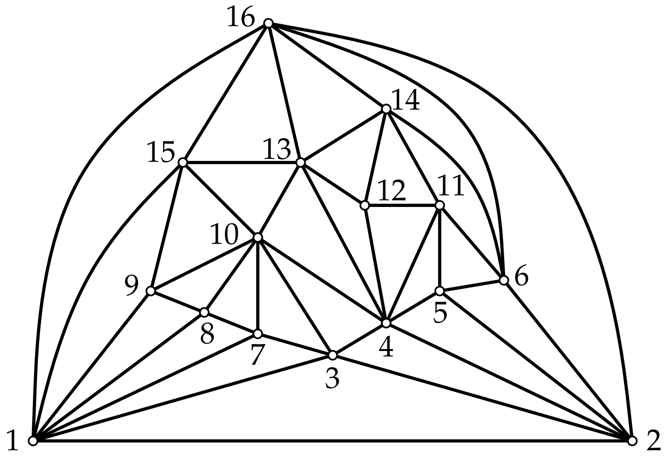


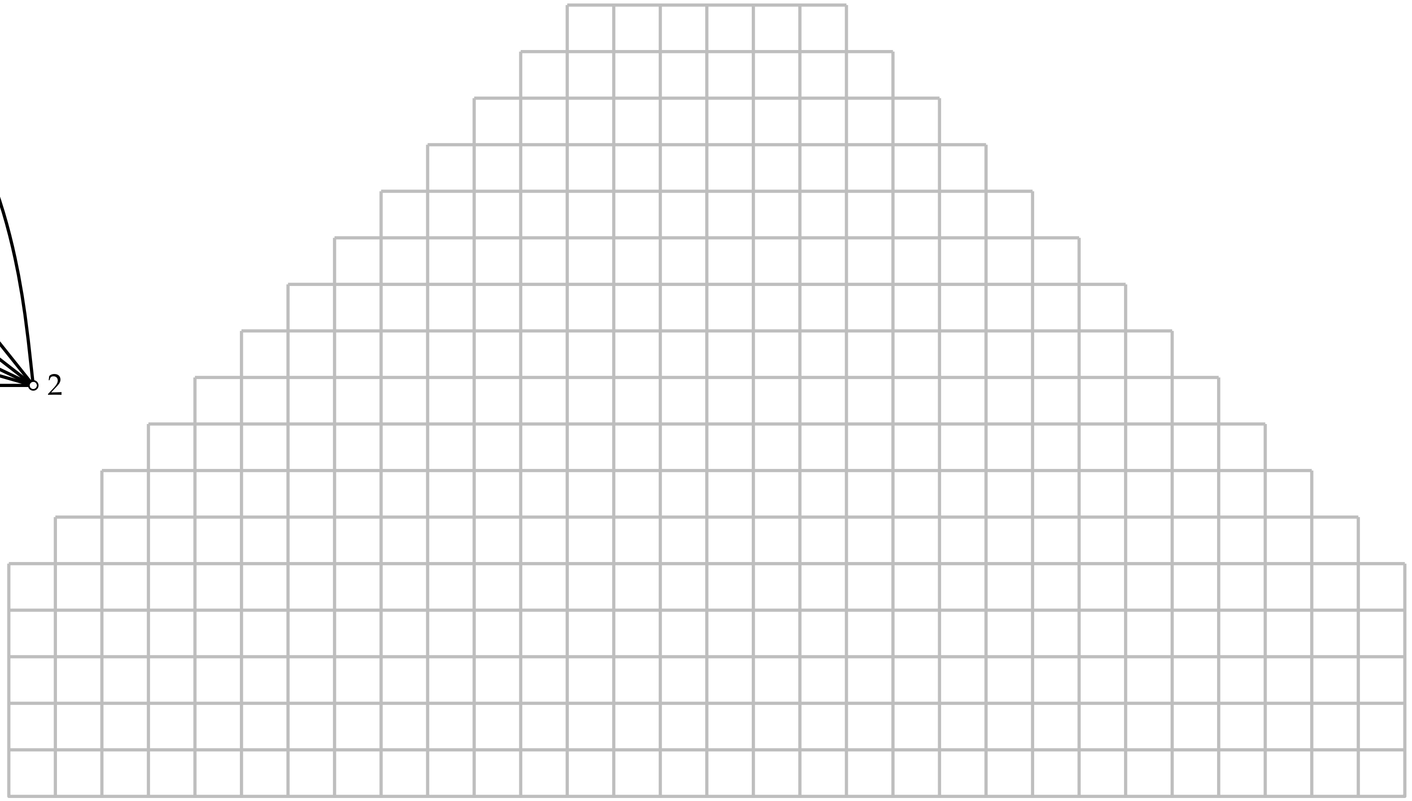
yes, beause w_p and w_q have even Manhattan distance



Shift Method – Example







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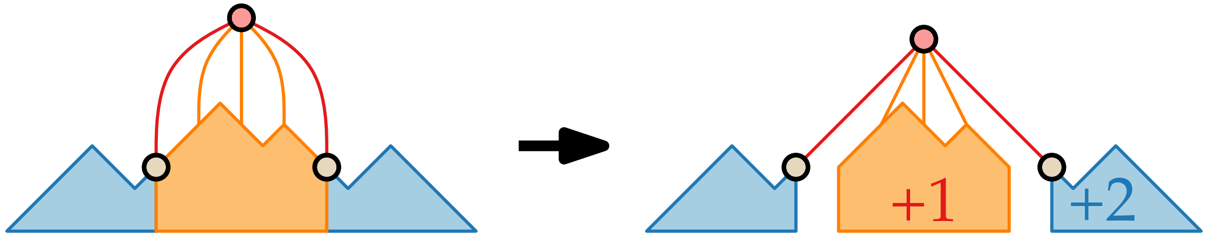

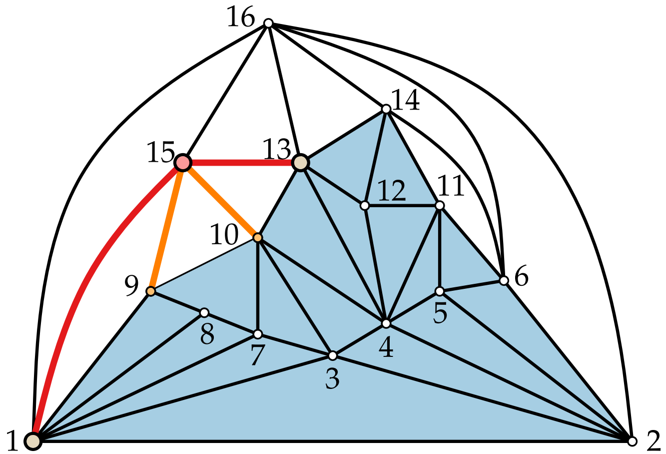

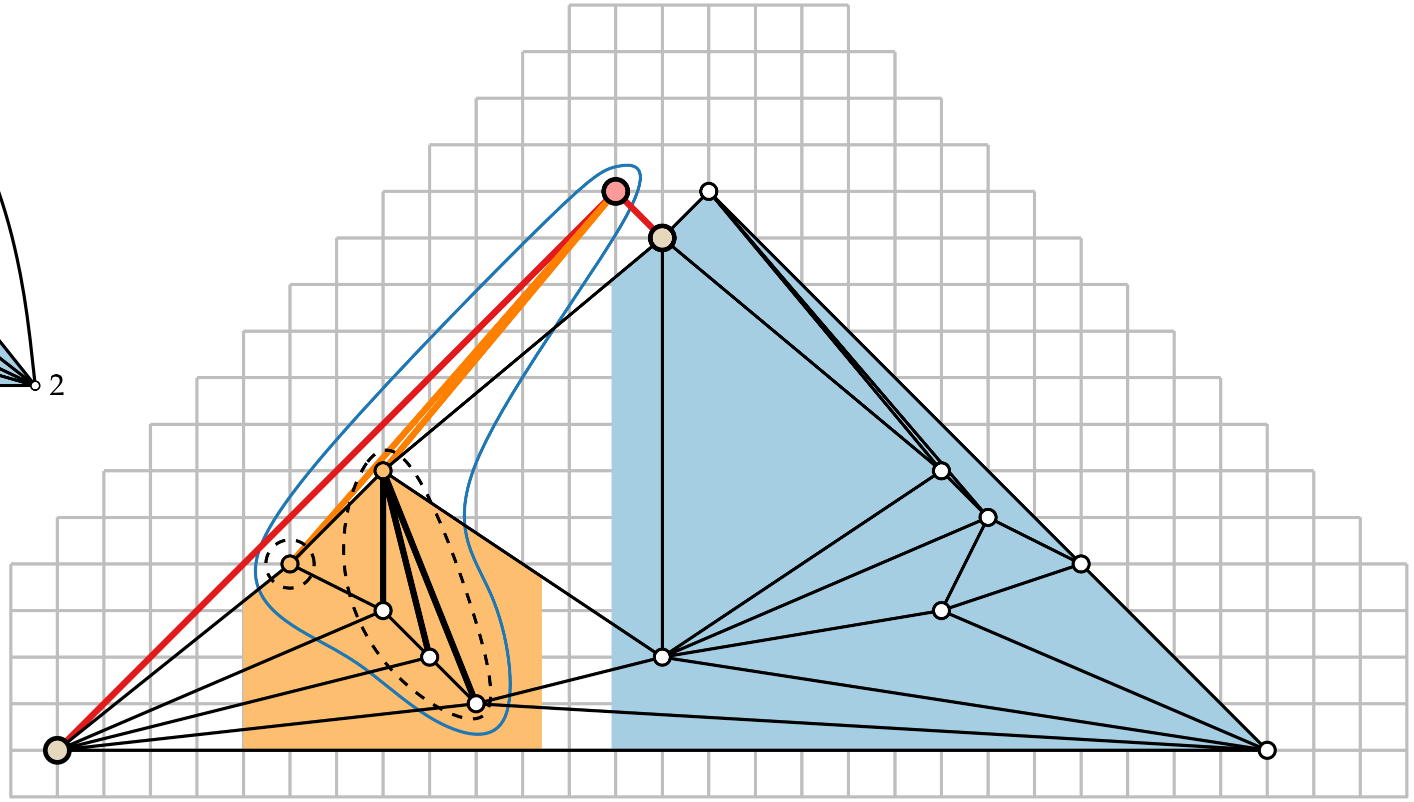

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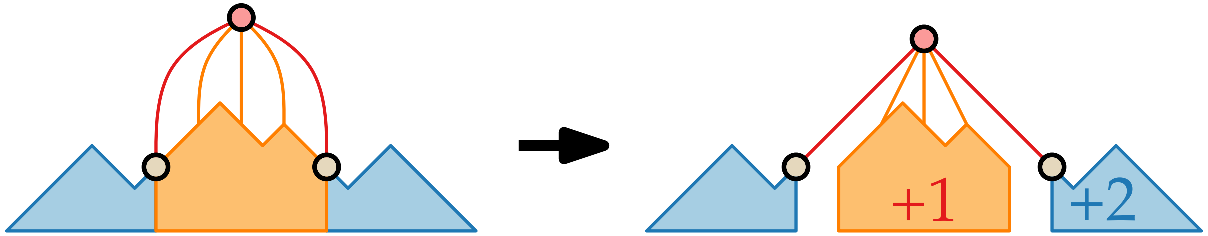
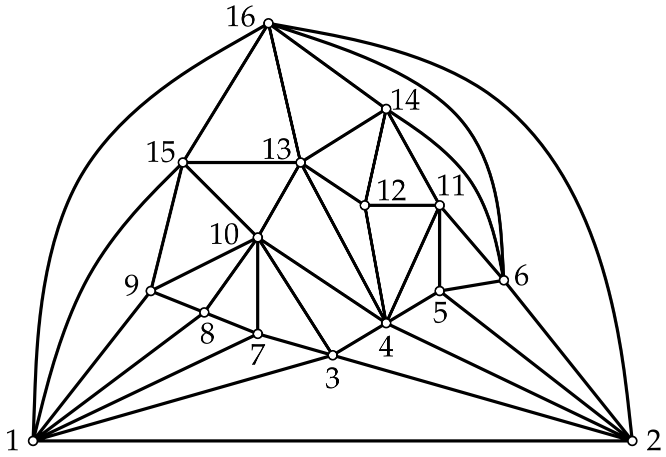
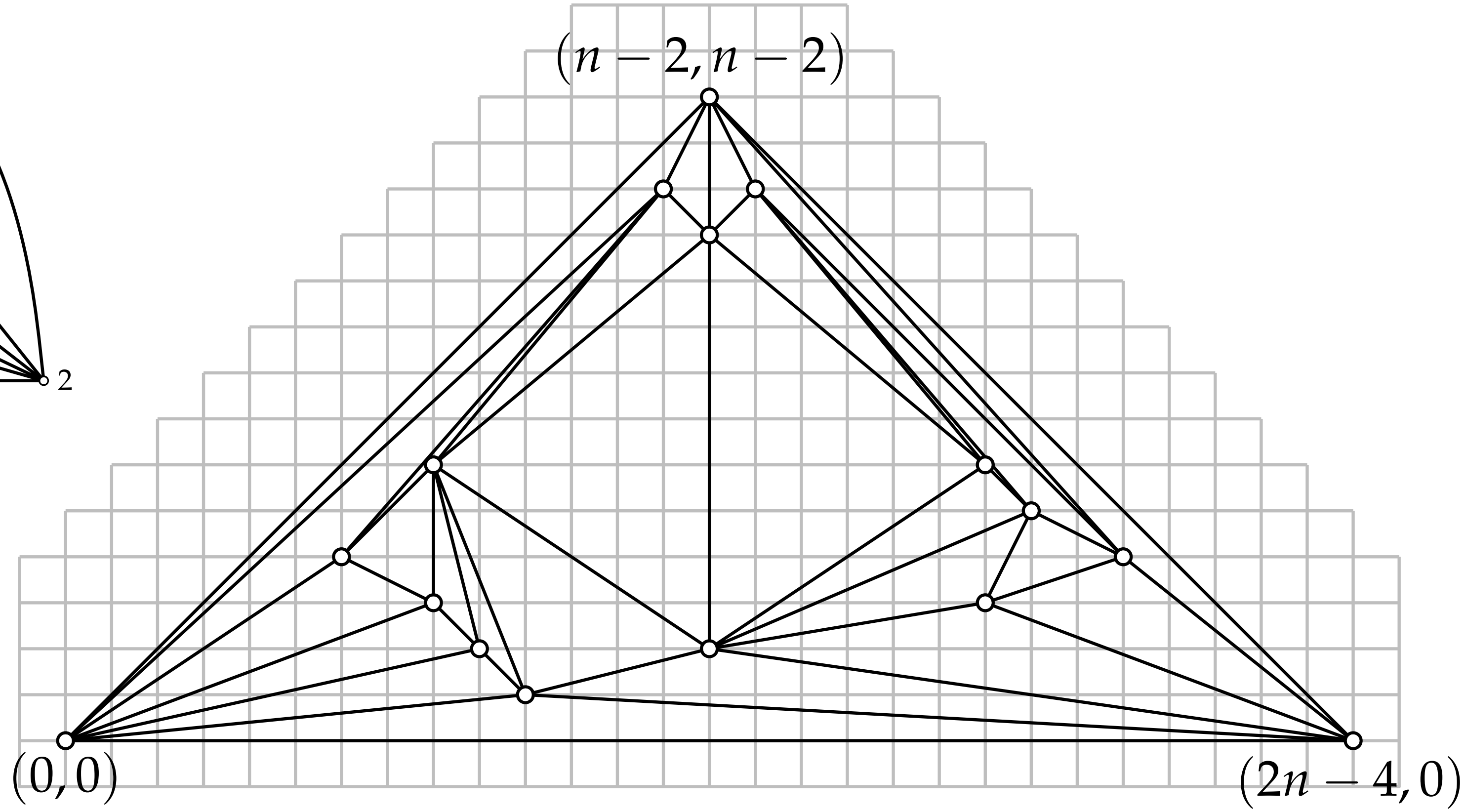




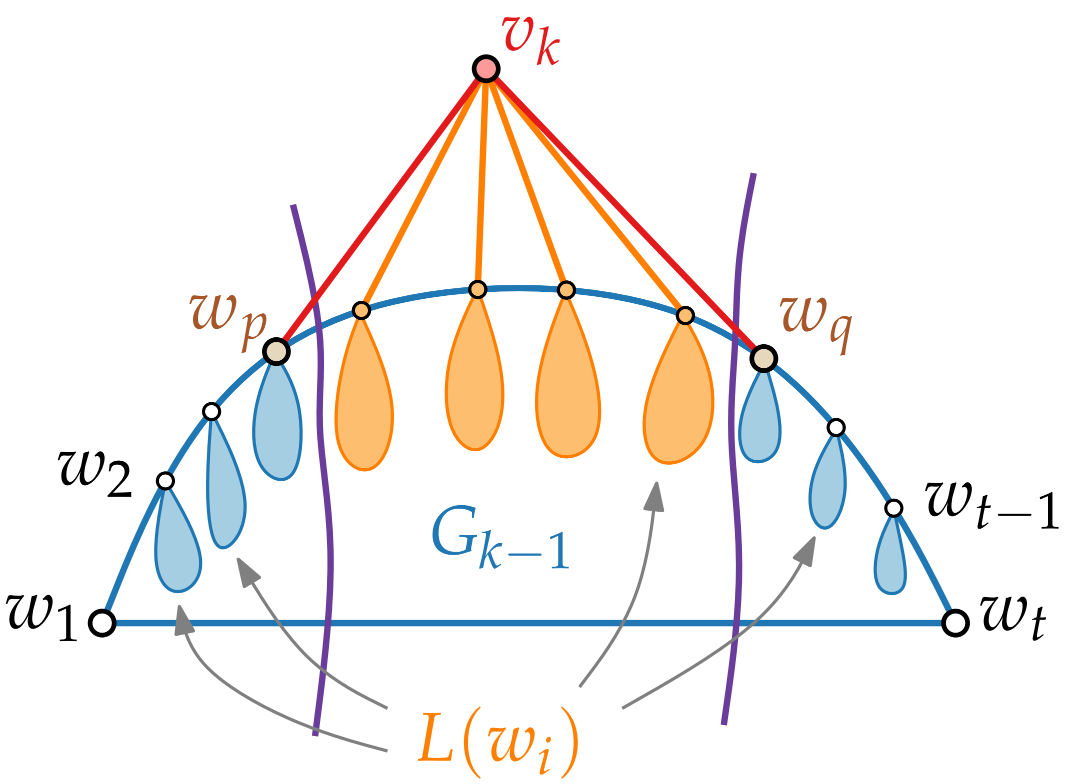


Shift Method – Planarity

Observations.

- Each internal vertex is covered exactly once.
- Covering relation defines a tree in G
- and a forest in G_i, $1 \leq i \leq n-1$.

Lemma.
Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even.



Shift Method – Planarity

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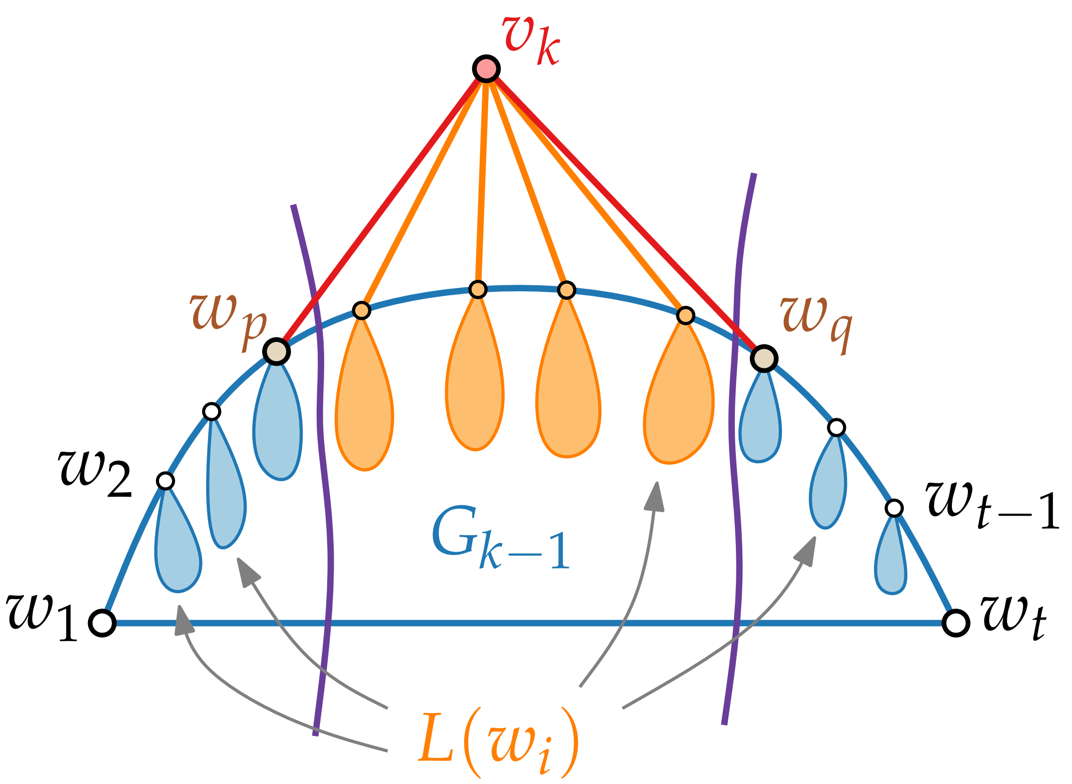
Lemma.
Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even. If we shift $L(w_i)$ by δ_i to the right, then we get a planar straight-line drawing.



Proof by induction:
If G_{k-1} is drawn planar and straight-line, then so is G_k.

Part V:
Linear Time

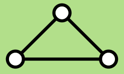
Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of G

for $i = 1$ to 3 **do**

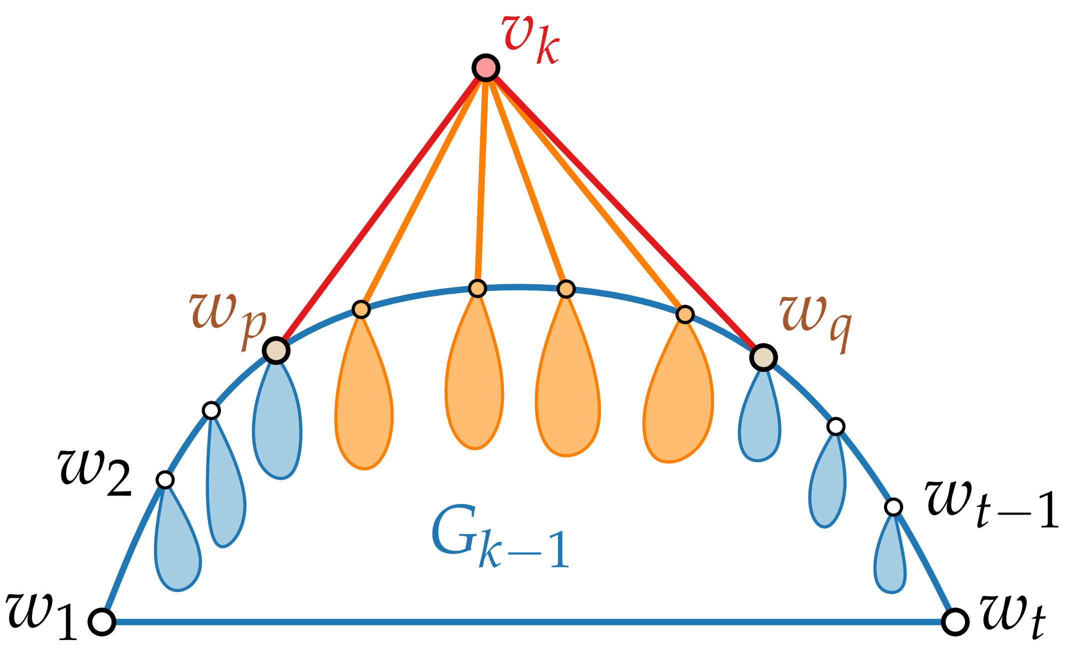
- $L(v_i) \leftarrow \{v_i\}$

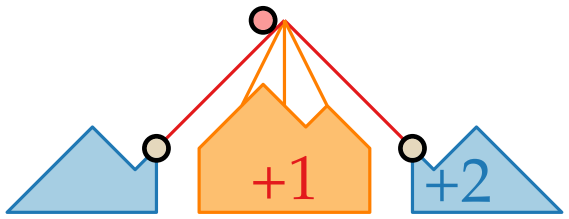
$P(v_1) \leftarrow (0,0); P(v_2) \leftarrow (2,0), P(v_3) \leftarrow (1,1)$



for $i = 4$ to n **do**

- Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1} and let $w_p, \ldots, w_q$ be the neighbours of v_i
- **for** $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do** // $\mathcal{O}(n^2)$ *in total*
  - $x(v) \leftarrow x(v) + 1$
- **for** $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do** // $\mathcal{O}(n^2)$ *in total*
  - $x(v) \leftarrow x(v) + 2$
- $P(v_i) \leftarrow$ intersection of $+1/-1$ diagonals through $P(w_p)$ and $P(w_q)$
- $L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$



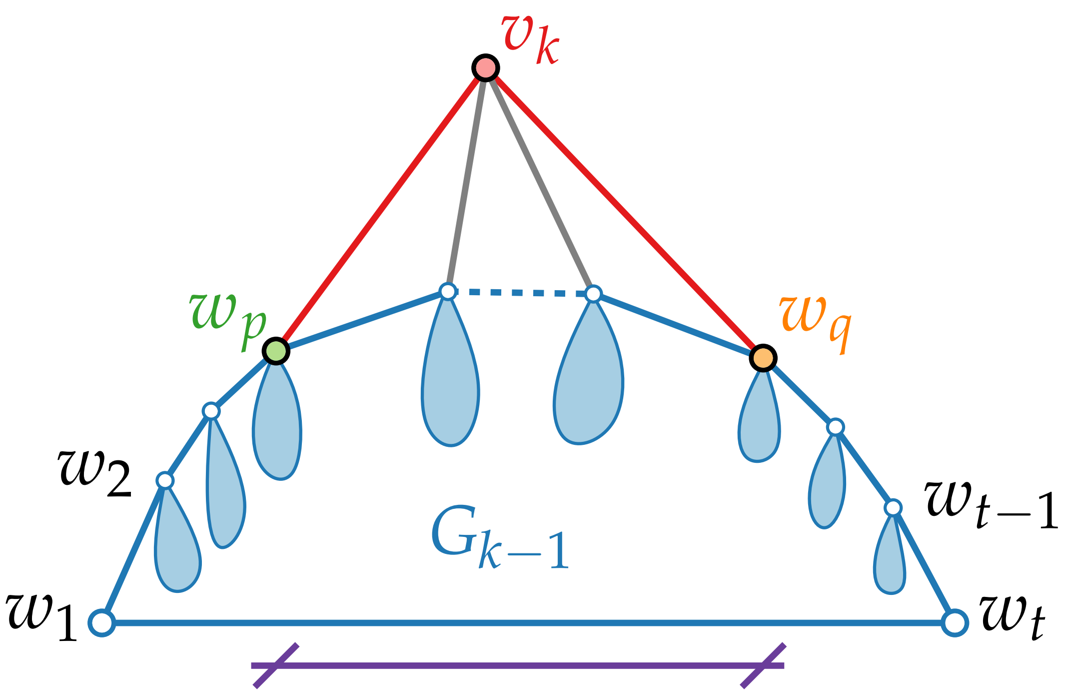


Running Time?

Shift Method – Linear Time Implementation

Idea 1.
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit x-coordinates,
we store x-distances.



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

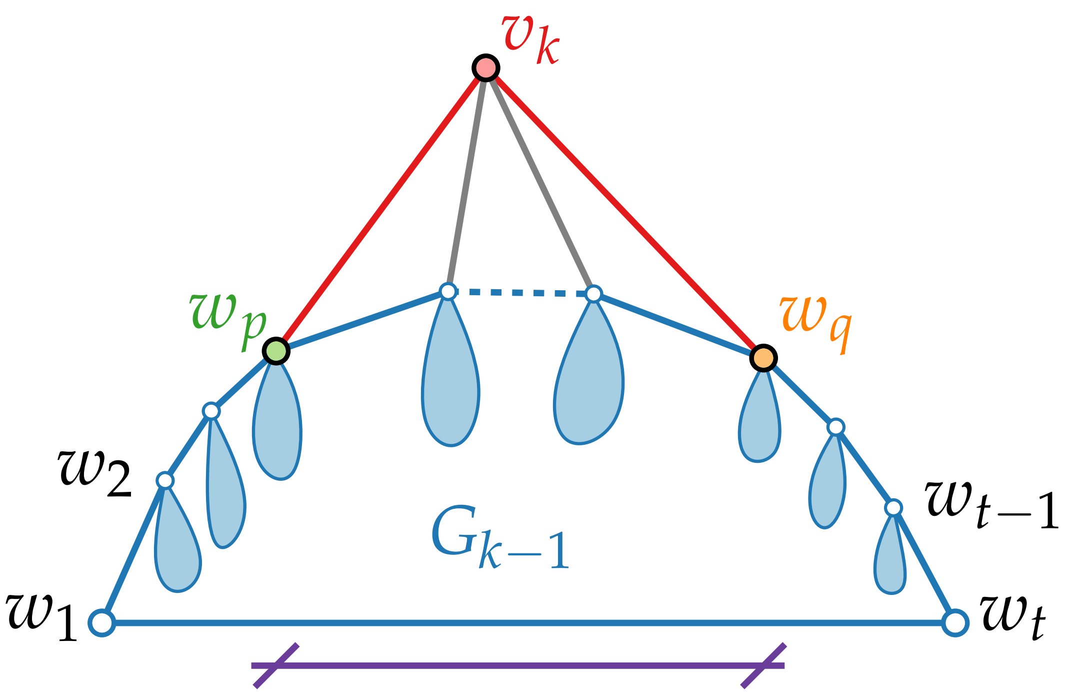
(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

Shift Method – Linear Time Implementation

Idea 1.
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit x-coordinates, we store x-distances.

After x distance for v_n computed, use preorder traversal to compute all x-coordinates.



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(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear Time Implementation

Relative x-distance tree.

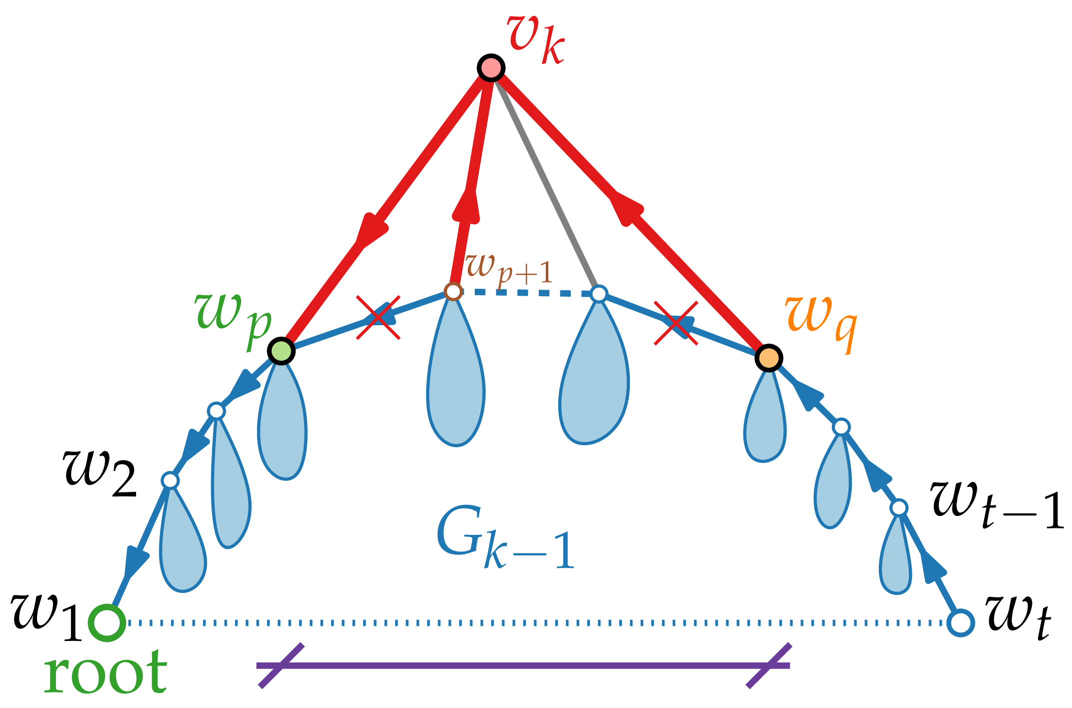
For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$

$\mathcal{O}(n)$ in total



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Result & Variations

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$. Such a drawing can be computed in $O(n)$ time.

Theorem. [Chrobak & Kant '97]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$ where all faces are drawn convex. Such a drawing can be computed in $O(n)$ time.

Theorem. [Brandenburg '08]
Every n-vertex planar graph has a planar straight-line drawing of size $\frac{4}{3}n \times \frac{2}{3}n$. Such a drawing can be computed in $O(n)$ time.