Visualization of Graphs

Lecture 1:
The Graph Visualization Problem

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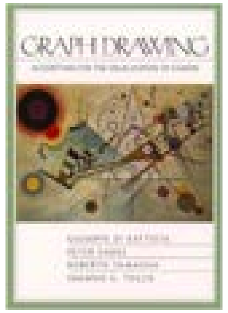
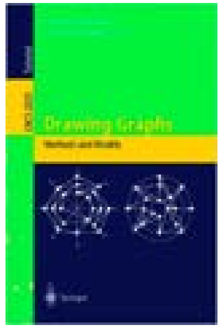
The slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, U. Trier

Part I:
Organizational & Overview

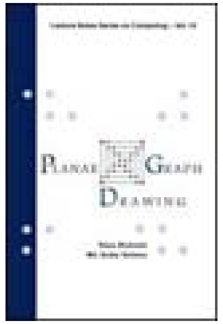
Books



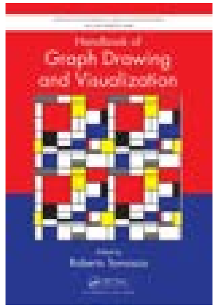
G. Di Battista, P. Eades, R. Tamassia, I. Tollis:
Graph Drawing: Algorithms for the Visualization of Graphs
Prentice Hall, 1998



M. Kaufmann, D. Wagner:
Drawing Graphs: Methods and Models
Springer, 2001



T. Nishizeki, Md. S. Rahman:
Planar Graph Drawing
World Scientific, 2004



R. Tamassia:
Handbook of Graph Drawing and Visualization
CRC Press, 2013
`http://cs.brown.edu/people/rtamassi/gdhandbook/`

What is this course about?

Topics

- Drawing Trees and Series-Parallel Graphs
- Straight-Line Drawings of Planar Graphs
- Orthogonal Grid Drawings
- Octilinear Drawings for Metro Maps
- Upwards Planar Drawings
- Hierarchical Layouts of Directed Graphs
- Contact Representations
- Visibility Representations
- The Crossing Lemma
- Beyond Planarity

Part II:
The Layout Problem

Graphs and their representations

Representation?

- Set notation

$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$
$E = \{\{v_1, v_2\}, \{v_1, v_8\}, \{v_2, v_3\}, \{v_3, v_5\}, \{v_3, v_9\}, \{v_3, v_{10}\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_9\}, \{v_5, v_8\}, \{v_6, v_8\}, \{v_6, v_9\}, \{v_7, v_8\}, \{v_7, v_9\}, \{v_8, v_{10}\}, \{v_9, v_{10}\}\}$

- Adjacency list

v_1: v_2, v_8
v_2: v_1, v_3
v_3: v_2, v_5, v_9, v_{10}
v_4: v_5, v_6, v_9
v_5: v_3, v_4, v_8
v_6: v_4, v_8, v_9
v_7: v_8, v_9
v_8: $v_1, v_5, v_6, v_7, v_9, v_{10}$
v_9: $v_3, v_4, v_6, v_7, v_8, v_{10}$
v_{10}: v_3, v_8, v_9

- Adjacency matrix

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
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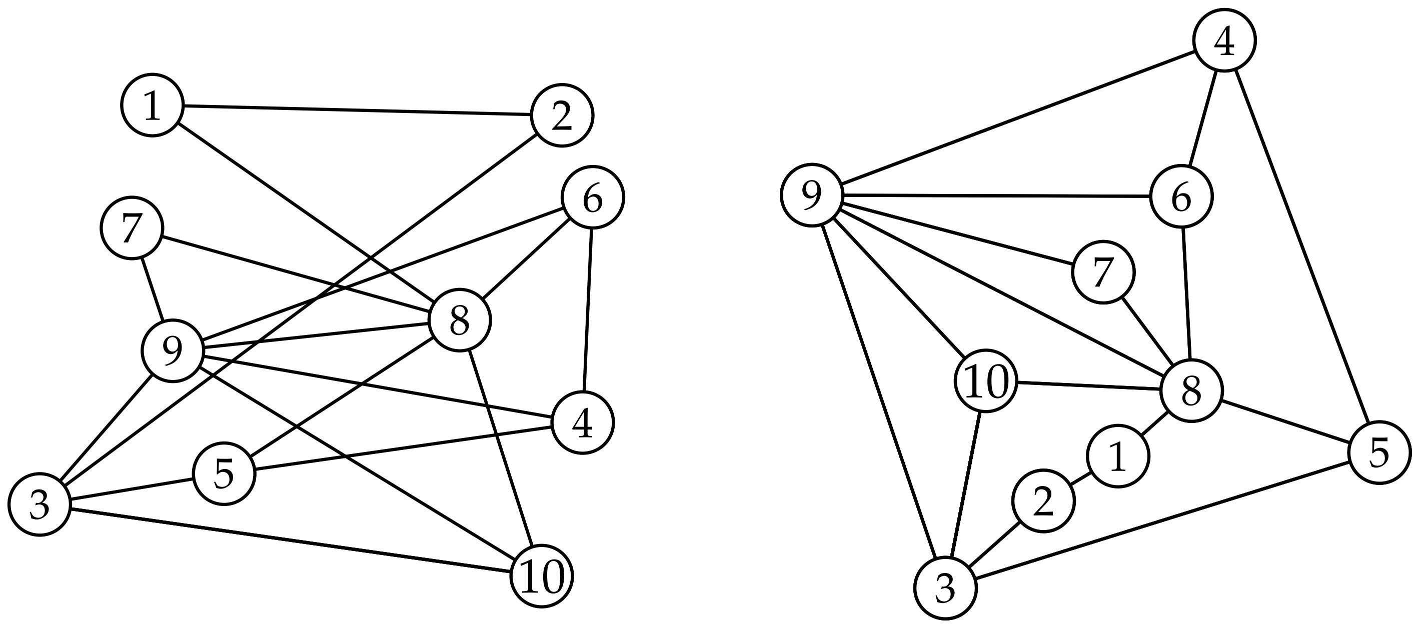
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}$$

What is a graph?

- graph $G = (V, E)$
- vertices $V = \{v_1, v_2, \ldots, v_n\}$
- edge $E = \{e_1, e_2, \ldots, e_m\}$

- Drawing



Why draw graphs?

Graphs are a mathematical representation of
real physical and abstract networks.

Abstract networks

- Social networks
- Communication networks
- Phylogenetic networks
- Metabolic networks
- Class/Object Relation Digraphs (UML)
- ...

Physical networks

- Metro systems
- Road networks
- Power grids
- Telecommunication networks
- Integrated circuits
- ...

Why draw graphs?

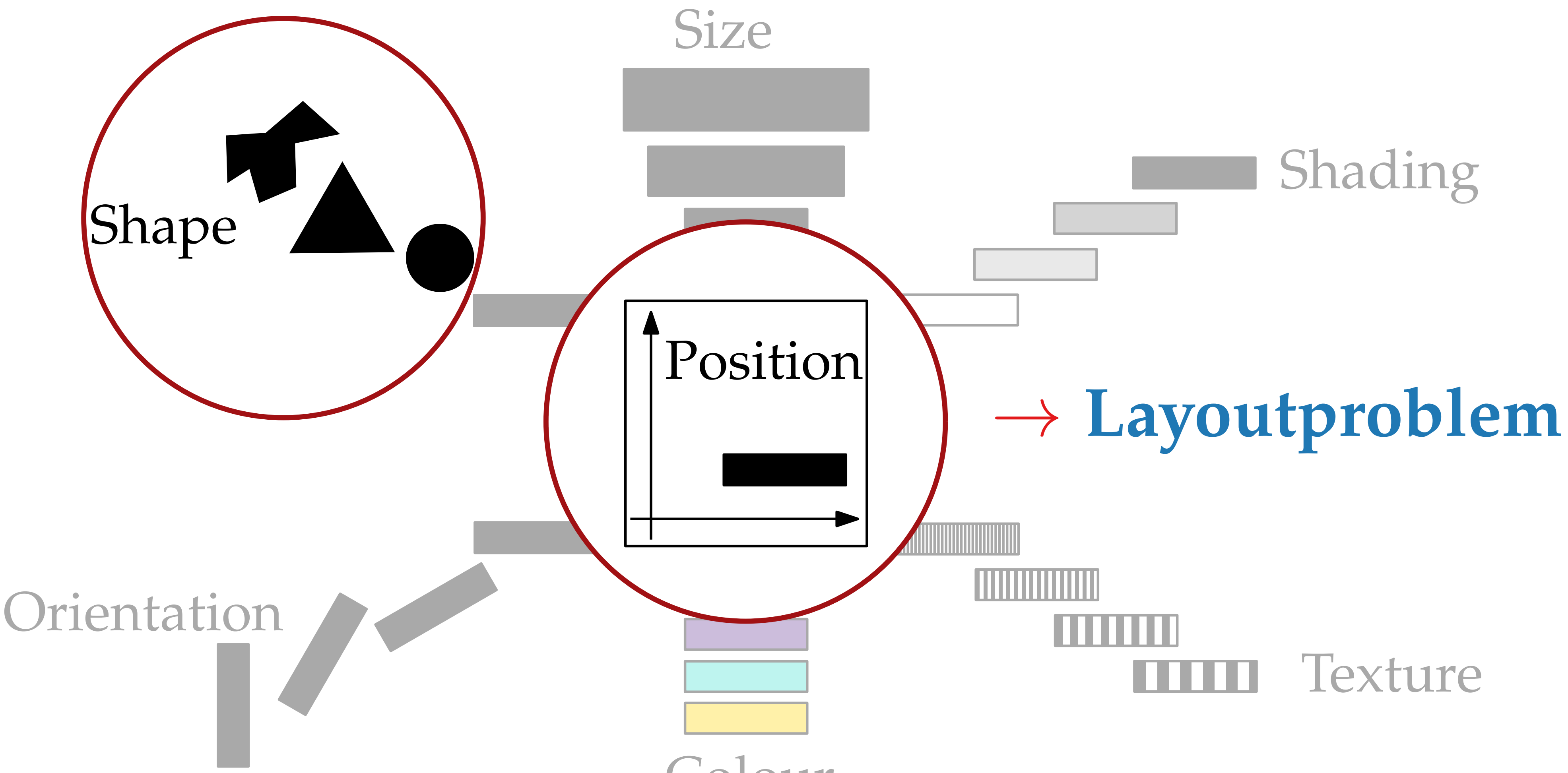
Graphs are a mathematical representation of real physical and abstract networks.

- **People think visually** – complex graphs are hard to grasp without good visualizations!
- Visualizations help with the **communication** and **explorati-on** of networks.
- Some graphs are too big to draw them by hand.

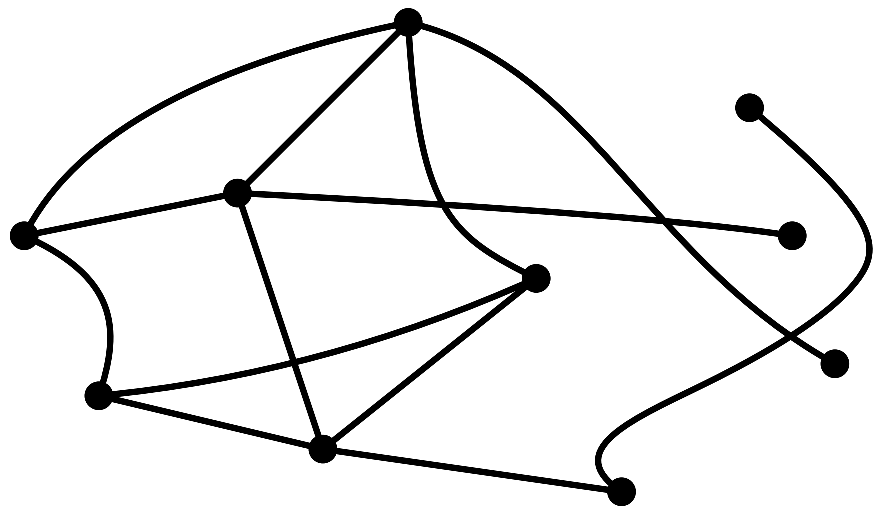
We need algorithms that draw graphs automatically to make networks more accessible to humans.

What are we interested in?

- Jacques Bertin defined visualising variables (1967)



The layout problem?



- *Here* restricted to the **standard representation**, so-called node-link diagrams.

Graph Visualization Problem

in: Graph $G = (V, E)$

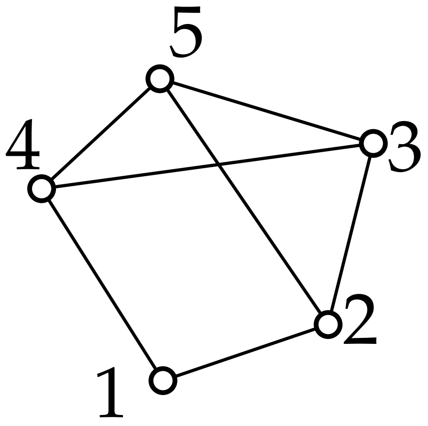
out: nice drawing Γ of G

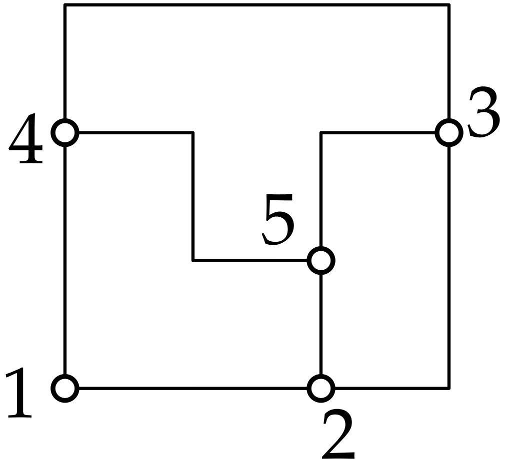
- $\Gamma \colon V \to \mathbb{R}^2$, vertex $v \mapsto$ point $\Gamma(v)$
- $\Gamma \colon E \to$ curves in $\mathbb{R}^2$, edge $\{u, v\} \mapsto$ simple, open curve $\Gamma(\{u, v\})$ with endpoints $\Gamma(u)$ und $\Gamma(v)$

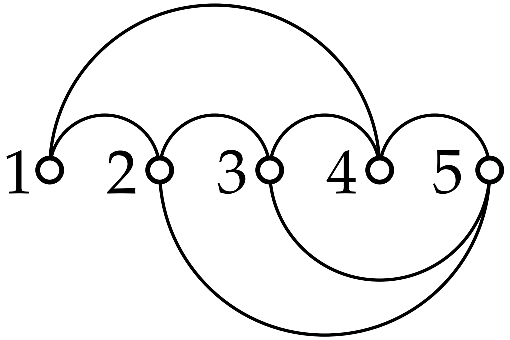
But what is a **nice** drawing?

Requirements of a graph layout

1. Drawing conventions and requirements, e.g.,
   - straight edges with $\Gamma(uv) = \overline{\Gamma(u)\Gamma(v)}$
   - orthogonal edges (i.e. with bends)
   - grid drawings
   - without crossings
2. Aesthetics to be optimized, e.g.
   - crossing/bend minimization
   - edge length uniformity
   - minimizing total edge length/drawing area
   - angular resolution
   - symmetry/structure
3. Local Constraints, e.g.
   - restrictions on neighboring vertices (e.g., “upward”).
   - restrictions on groups of vertices/edges (e.g., “clustered”).







→ lead to NP-hard optimization problems

→ such criteria are often inversely related

The layout problem

Graph Visualization Problem

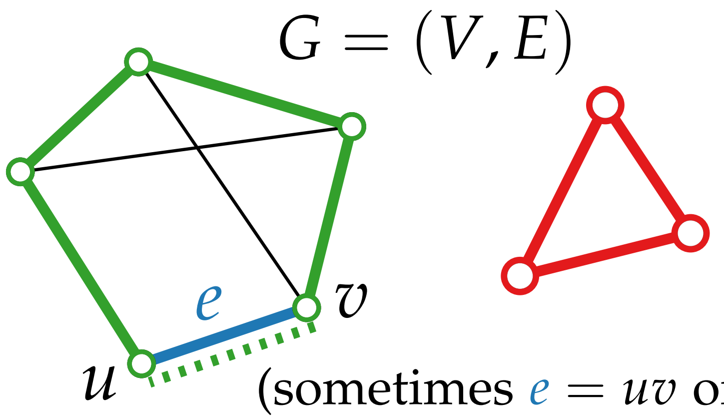
in: Graph $G = (V, E)$

out: Drawing Γ of G such that

- **drawing conventions** are met,
- **aesthetic criteria** are optimised, and
- some **additional constraints** are satisfied.

Part III:
Basics

Basic Definitions



(sometimes $e = uv$ or $e = (u, v)$)

Edge $e = \{u, v\} \in E$:

- e **incident** to u and v
- u, v **end vertices** of e
- u **adjacent** to v
- u and v are **neighbors**

degree $\deg(v)$:
number of edges incident to v

u-v-path of length ℓ:
Sequence of $\ell + 1$ distinct adjacent vertices (and ℓ connecting edges), starting with u and ending with v:
$u - \{u, v_1\} - v_1 - \cdots - v_{\ell-1} - \{v_{\ell-1}, v\} - v$

simple cycle: u-u-path

connected: There is a u-v-path for every $u, v \in V$

v **reachable** from u: There is a u-v-path

subgraph: graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$

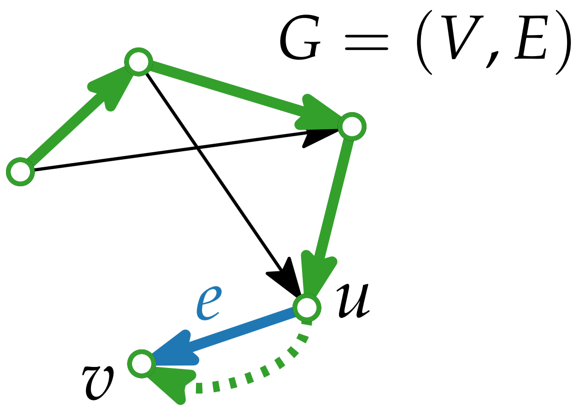
induced subgraph: subgraph with $E' = \binom{V'}{2} \cap E$

connected component: maximal connected subgraph

Handshaking-Lemma.
$\sum_{v \in V} \deg(v) = 2|E|$

Corollary.
The number of odd-degree vertices is even.

Directed Graphs



Edge $e = (u, v) \in E$:

- u is **source** of e
- v is **target** of e

indegree $\deg^-(v)$:
number of edges for which v is the target

outdegree $\deg^+(v)$:
number of edges for which v is the source

Handshaking-Lemma.
$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$

directed u-v-path:
$u - (u, v_1) - v_1 - \cdots - v_{\ell-1} - (v_{\ell-1}, v) - v$

directed cycle: directed u-u-path

acyclic: no directed cycles

connected: There is a directed u-v-path or v-u-path for every $u, v \in V$

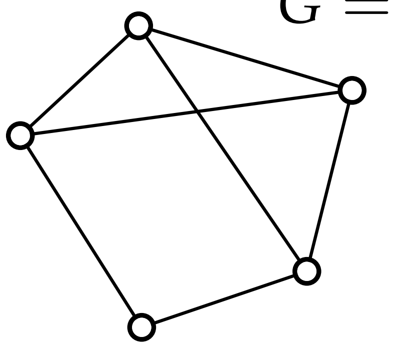
v **reachable** from u: There is a directed u-v-path

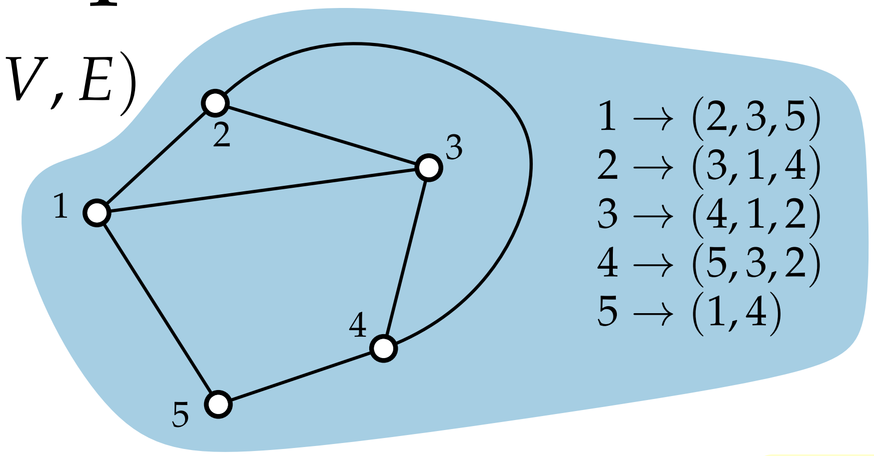
~~connected component~~

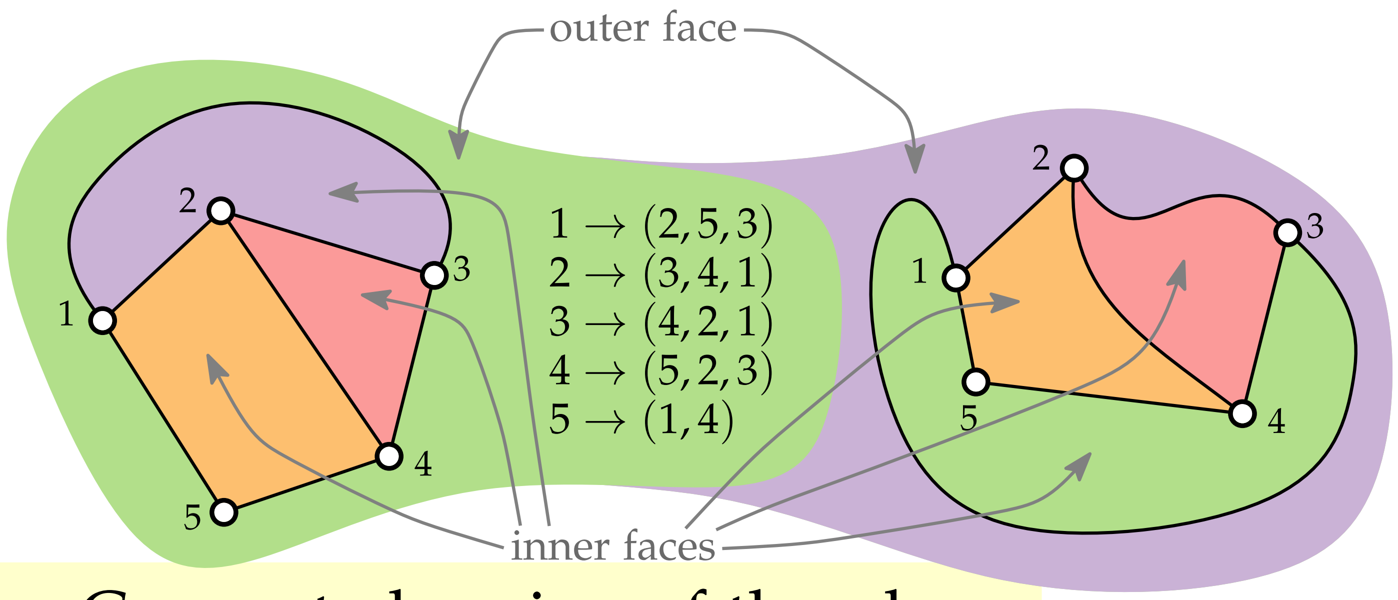
Part IV: Planarity

Planar Graphs

$G = (V, E)$







G is **planar**:
it can be drawn in such a way that no edges cross each other.

planar embedding:
Clockwise orientation of adjacent vertices around each vertex.

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

faces: Connected region of the plane bounded by edges

Euler's polyhedra formula.

#faces - #edges + #vertices = #conn.comp. + 1

$$f - m + n = c + 1$$

Proof. By induction on m:

$m = 0 \Rightarrow f = 1$ and $c = n$

$\Rightarrow 0 - 0 + c = c + 1$ ✓

$m > 1 \Rightarrow$ remove 1 edge $e \Rightarrow m - 1$

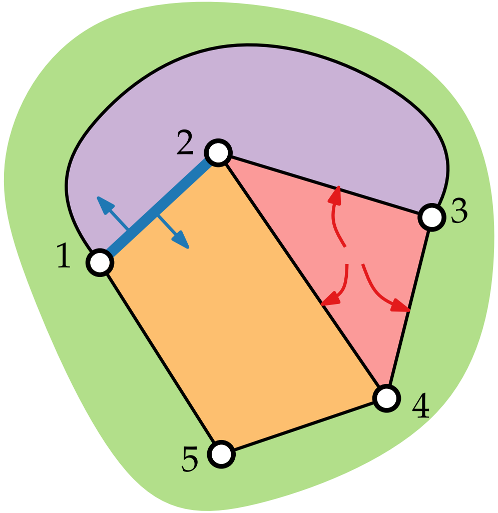
$\Rightarrow c + 1$

$\Rightarrow f + 1$

Properties of Planar Graphs

Euler's polyhedra formula.
#faces - #edges + #vertices = #conn.comp. + 1
f - m + n = c + 1

Theorem. G simple planar graph with $n \geq 3$.
1. $m \leq 3n - 6$ 2. $f \leq 2n - 4$
3. There is a vertex of degree at most five



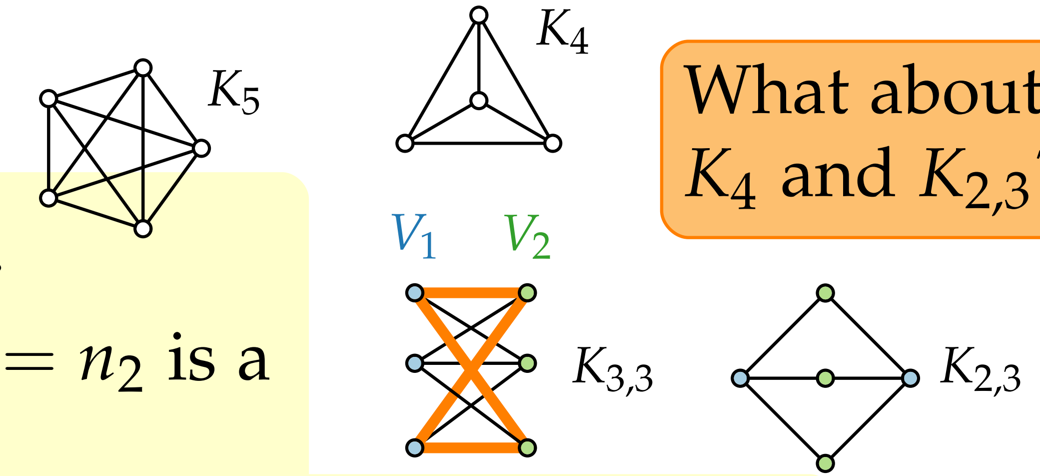
Proof. 1. Every edge incident to ≤ 2 faces
Every face incident to ≥ 3 edges
$\Rightarrow 3f \leq 2m$
$\Rightarrow 6 \leq 3c + 3 \leq 3f - 3m + 3n \leq 2m - 3m + 3n = 3n - m$
$\Rightarrow m \leq 3n - 6$

2. $3f \leq 2m \leq 6n - 12 \Rightarrow f \leq 2n - 4$

3. $\sum_{v \in V} \deg(v) = 2m \leq 6n - 12$
$\Rightarrow \min_{v \in V} \deg(v) \leq 1/n \sum_{v \in V} \deg(v) < 6$

Handshaking-Lemma.
$\sum_{v \in V} \deg(v) = 2|E|$

Complete graphs



What about K_4 and $K_{2,3}$?

$K_n = \left(V, \binom{V}{2}\right)$ is the **complete** graph on n vertices.

$K_{n_1,n_2} = (V_1 \cup V_2, V_1 \times V_2)$ with $|V_1| = n_1$ and $|V_2| = n_2$ is a **complete bipartite** graph on $n = n_1 + n_2$ vertices.

A **bipartite** graph is a subgraph of a K_{n_1,n_2}; V_1 and V_2 are called **bipartitions**.

Theorem. K_5 and $K_{3,3}$ are not planar.

Theorem. G simple planar graph with $n \geq 3$.
1. $m \leq 3n - 6$
2. $f \leq 2n - 4$
3. There is a vertex of degree at most five

Theorem. G simp. pl. **bipartite** graph, $n \geq 3$.
1. $m \leq 2n - 4$
2. $f \leq n - 2$
3. There is a vertex of degree at most three

Proof.

K_5: $m = \binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10 > 9 = 3 \cdot 5 - 6 = 3n - 6$ ✓

$K_{3,3}$: $m = 3 \cdot 3 = 9 < 12 = 3 \cdot 6 - 6 = 3n - 6$

$\Rightarrow$ *no* contradiction to the theorem!

There is no cycle of length 3.

Every face incident to ≥ 4 edges (in hypothetical planar drawing)

$\Rightarrow 4f \leq 2m$

$\Rightarrow 8 \leq 4c + 4 \leq 4f - 4m + 4n \leq 2m - 4m + 4n = 4n - 2m$

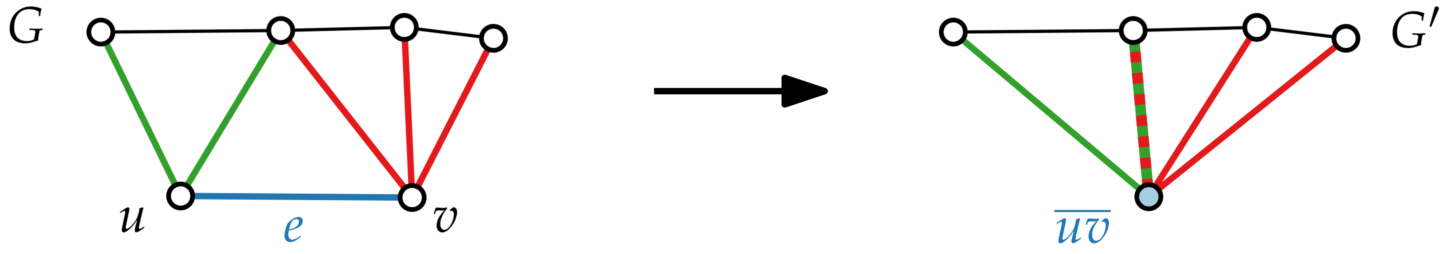
$\Rightarrow m \leq 2n - 4 = 2 \cdot 6 - 4 = 8 < 9 = m$ ✓

Contractions and Minors

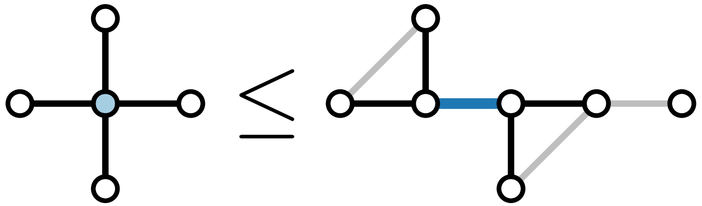
G simple graph and $e = uv \in E$

Contracting e gives the graph $G' = (V', E')$
$V' = V \setminus \{u, v\} \cup \overline{uv}$
$E' = E \setminus (\bigcup_{w \in V} \{uw, vw\}) \cup \bigcup_{x \in \mathrm{Adj}(u) \cup \mathrm{Adj}(v)} \overline{uv}x$

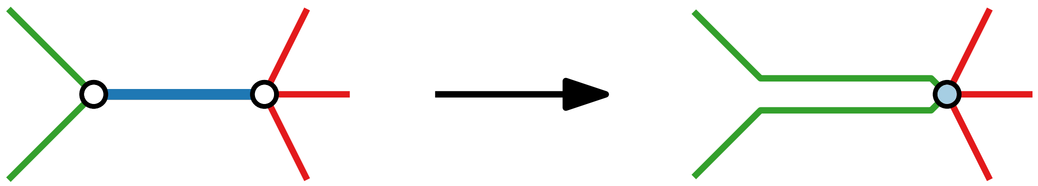
(multi-edges are merged)



A graph H is a **minor** of G (write $H \leq G$)
if it is obtained by a set of contractions
from a subgraph of G.



Observation.
G planar, $H \leq G \Rightarrow H$ planar



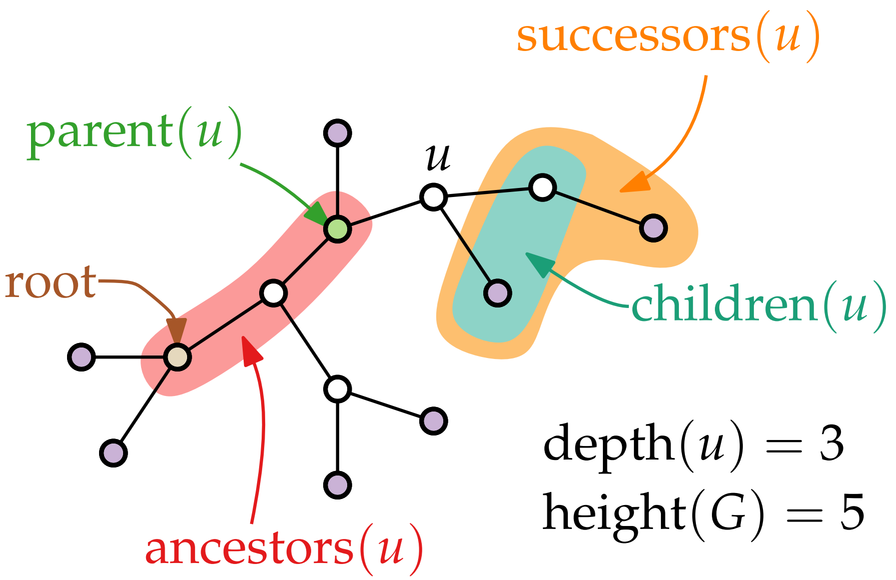
Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
neither K_5 nor $K_{3,3}$ minor of G

Kazimierz Kuratowski
Warschau 1896–1980 Warschau

Part V:
Binary Search Trees

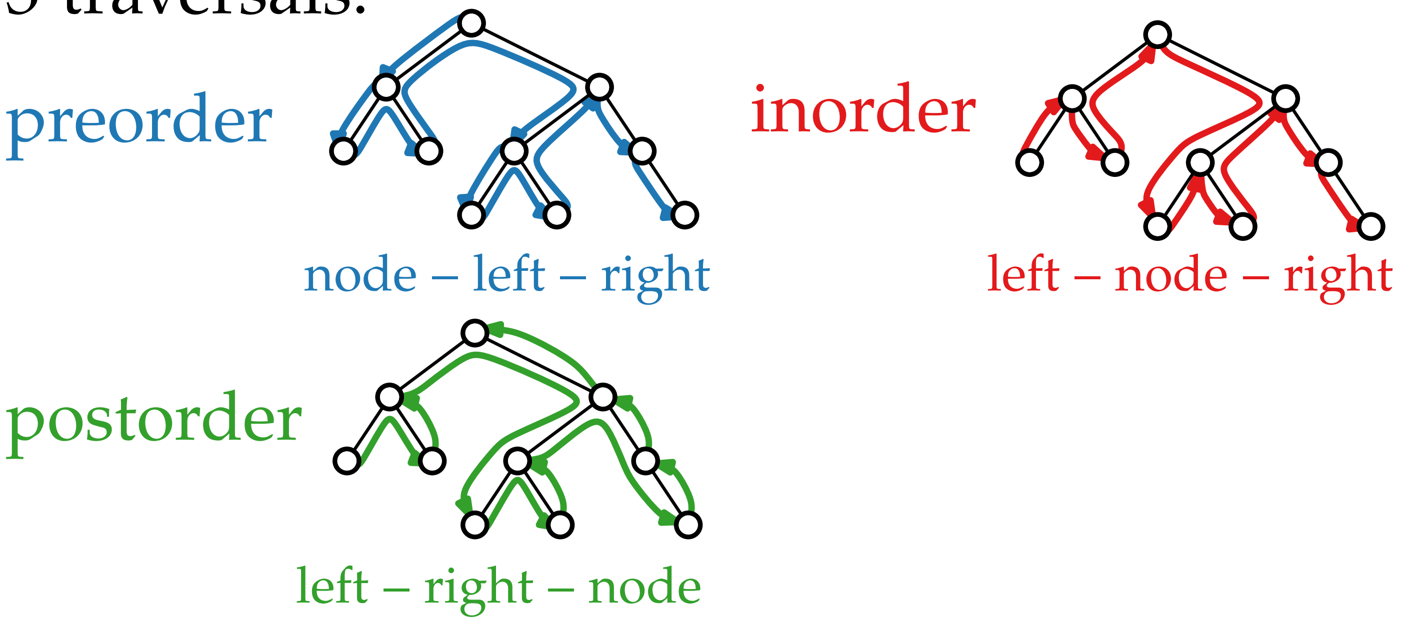
(Rooted) Trees

G is a **tree** if the following equivalent conditions hold:
1. there is exactly one v-w-path between any $v, w \in V$
2. G cycle-free and connected
3. G cycle-free and $m = n - 1$
4. G connected and $m = n - 1$



Leaf: Vertex of degree 1
Rooted tree: tree with designated **root**
Ancestor: Vertex on path to root
Parent: Neighbor on path to root
Successor: Vertex not on path to root
Child: Neighbor not on path to root
Depth: Length of path to root
Height: Maximum depth of a leaf

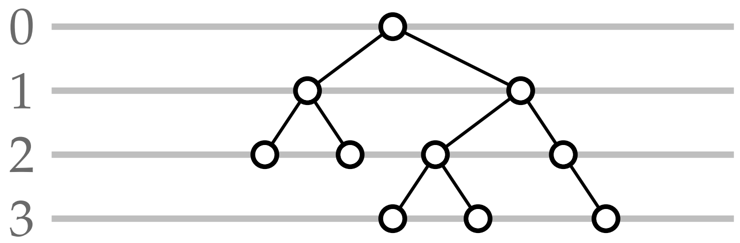
3 traversals:



Binary Tree: At most two children per vertex (left / right child)

First Grid Layout of Binary Trees

1. Choose y-coordinates: $y(u) = \text{depth}(u)$



2. Choose x-coordinates: